Reciprocal Square Root

Accelerated using Mixed Hardware and Software Techniques

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Inria
KALRAY THE POWER OF MORE

Context

Reciprocal (and) Square Root Function(s)

To accelerate ML, we need some elementary functions. We are specifically interested in:

- $\frac{1}{\sqrt{x}}$ for batch normalisation
- $\frac{1}{x}$ for various activation functions (softmax $\frac{\exp^{x_i}}{\sum_k \exp^{x_k}}$, sigmoid $\frac{1}{1+\exp^{-x}}$, swish $\frac{x}{1+\exp^{-x}}$, tanh $\frac{1-\exp^{-2x}}{1+\exp^{-2x}}$,...)
- $\sqrt{x}$ in case anyone invents yet another activation function between now and when the accelerator is produced

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Target format

- FP16, typical in ML accelerator, probably precise enough
- FP32, just in case (and handy for other applications).

Generalised Newton-Raphson iterations

Big picture

1. Guess the result.

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Goal: Not spend too much time on step 2 (and 3).

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Hardware: Seed table

We need a very good guess.
Guess is pre-computed, and stored in hardware in a table.

A very good guess is an expensive guess. . .

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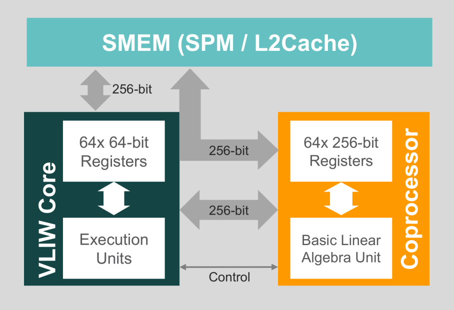
Adapting table compression and function evaluation techniques

Why not do everything in hardware ?

Kalray MPPA



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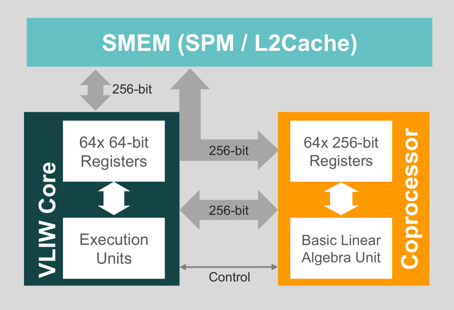
- Core can do either FP32 $\frac{1}{x}$ or FP32 $\sqrt{x}$: fully in hardware, pipelined in 12 cycles.



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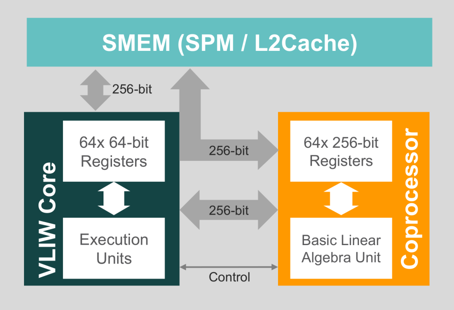

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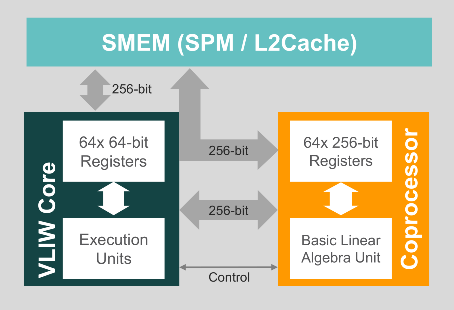

→ Use the full power of the FMAs to compute the iterations.
Decent throughput, small hardware footprint.

The expected efficiency

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Fun	Measure	FP32			FP16	
		Core (CR)	Algo (CR)	Algo (FR)	Algo (CR)	Algo (err $<$ 1.4ulp)
$\frac{1}{x}$	Latency	12	21	13	13	5
	Throughput	1	3.2	13	5.3	16
$\sqrt{x}$	Latency	12	22	14	13	5
	Throughput	1	2.3	4	4	16
$\frac{1}{\sqrt{x}}$	Latency	24	$\emptyset$	14	1	1 (CR)
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→ Always better throughput, and very fast FP16

How to build this Seed table ?

A guess for $\frac{1}{\sqrt{x}}$ for every FP32 number?

2^{31} guesses of size 31

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Argument Reduction for $\frac{1}{\sqrt{x}}$

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{2^e \times 1.F}} = \begin{cases} 2^{-k} \times \frac{1}{\sqrt{1.F}} & \text{if } e = 2k, k \in \mathbb{Z} \\ 2^{-k} \times \frac{1}{\sqrt{2 \times 1.F}} & \text{if } e = 2k+1, k \in \mathbb{Z} \end{cases}$$

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We just need fixed-point guesses for $\frac{1}{\sqrt{1.F}}$ and $\frac{1}{\sqrt{2\times 1.F}}$, for $F \in \mathrm{uFix}(-1, -23)$.

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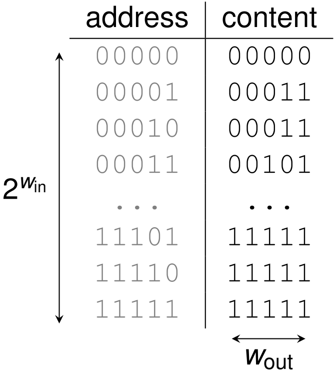

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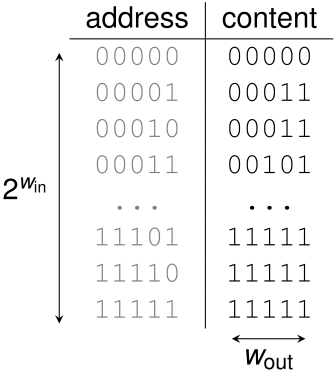

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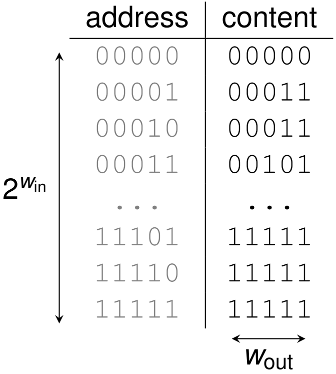

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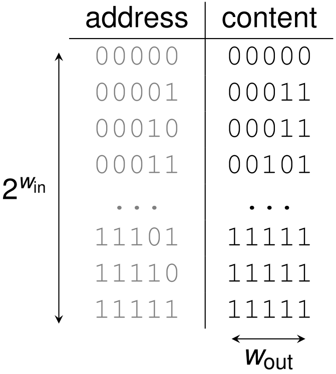

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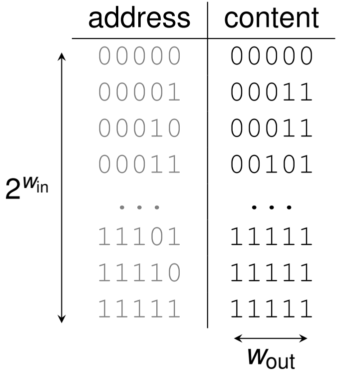

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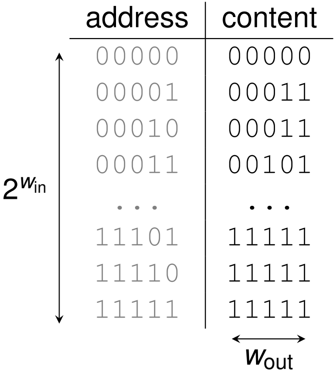

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How do we fill the table?

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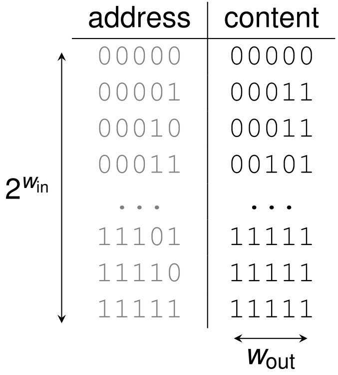
3. If it does not work, try another seed.



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Compute the parameters the seed table

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Theory says $w_{\text{in}} = w_{\text{out}} = 12$ should work.

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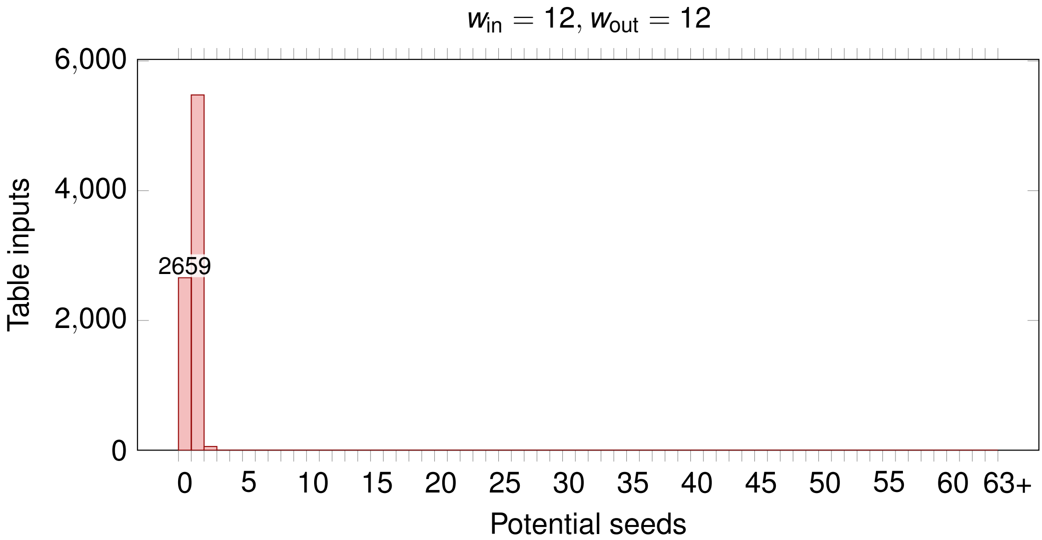

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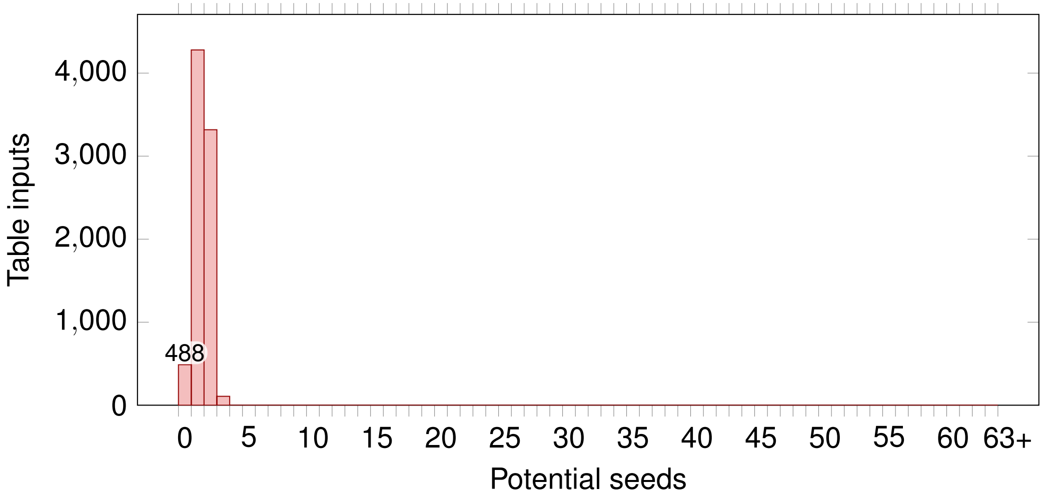
$w_{\text{in}} = 12, w_{\text{out}} = 13$



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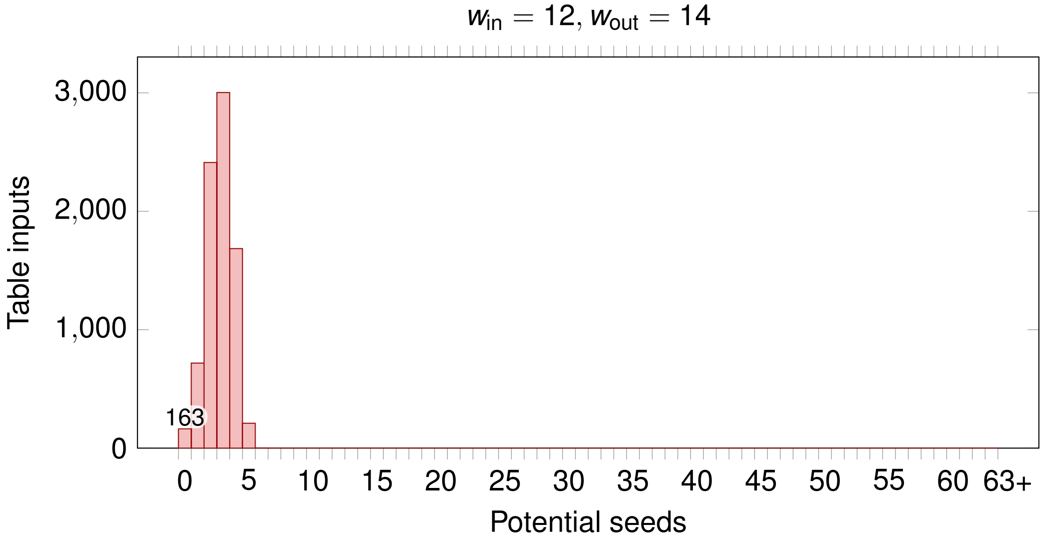

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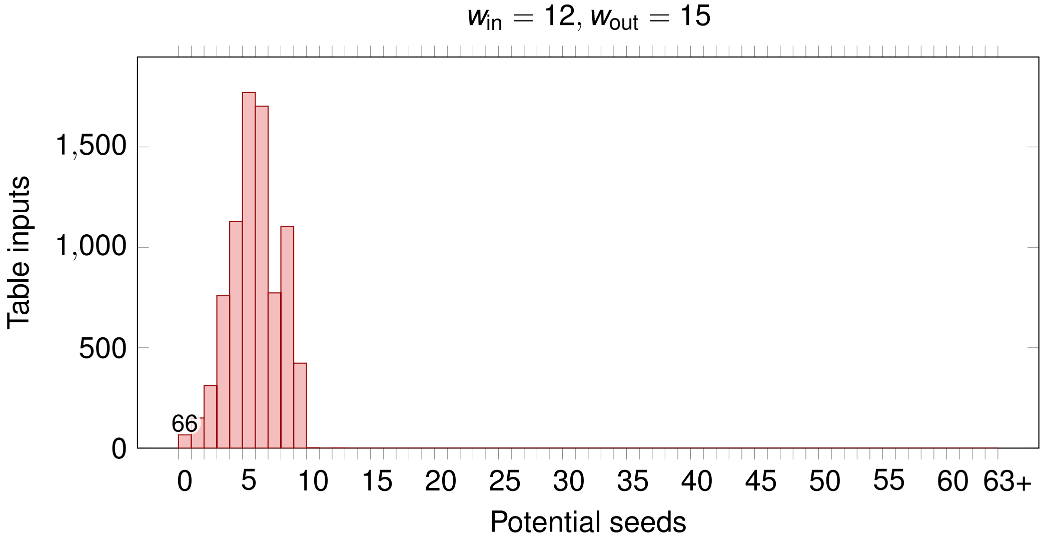

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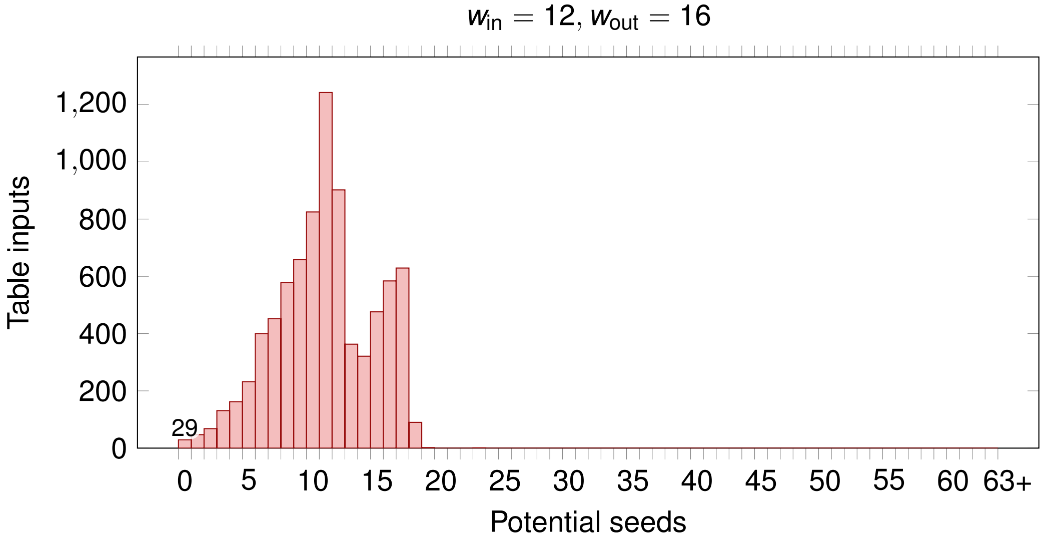

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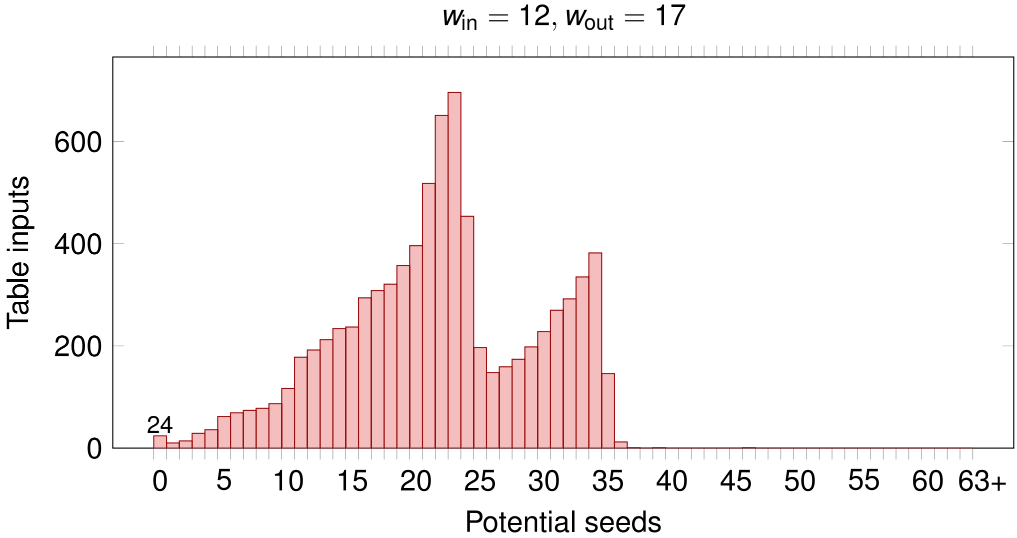

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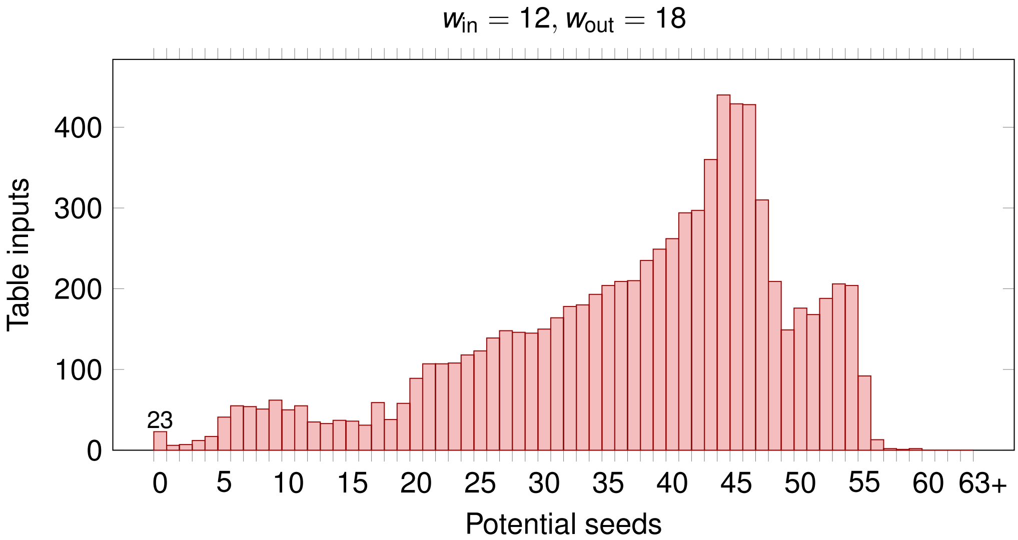

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Something isn’t working…

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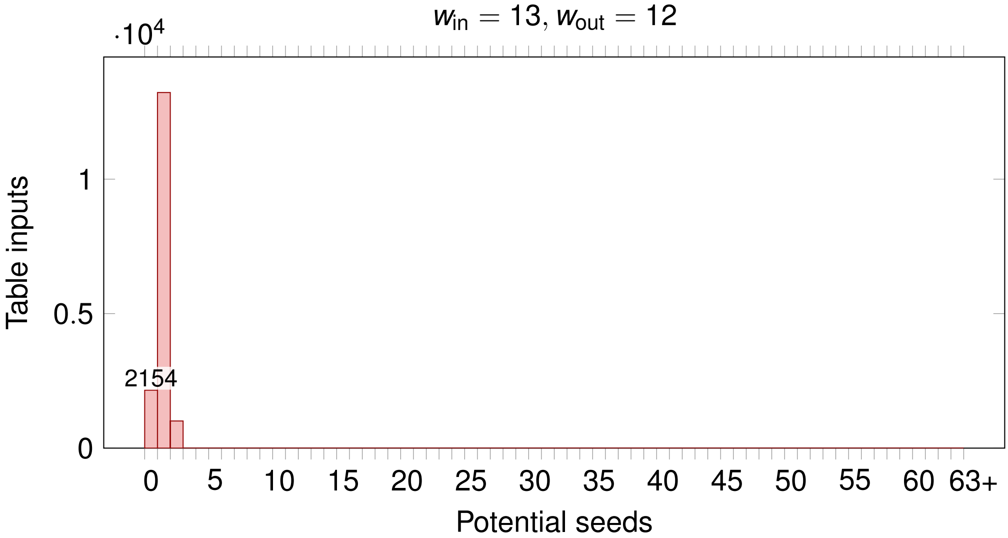

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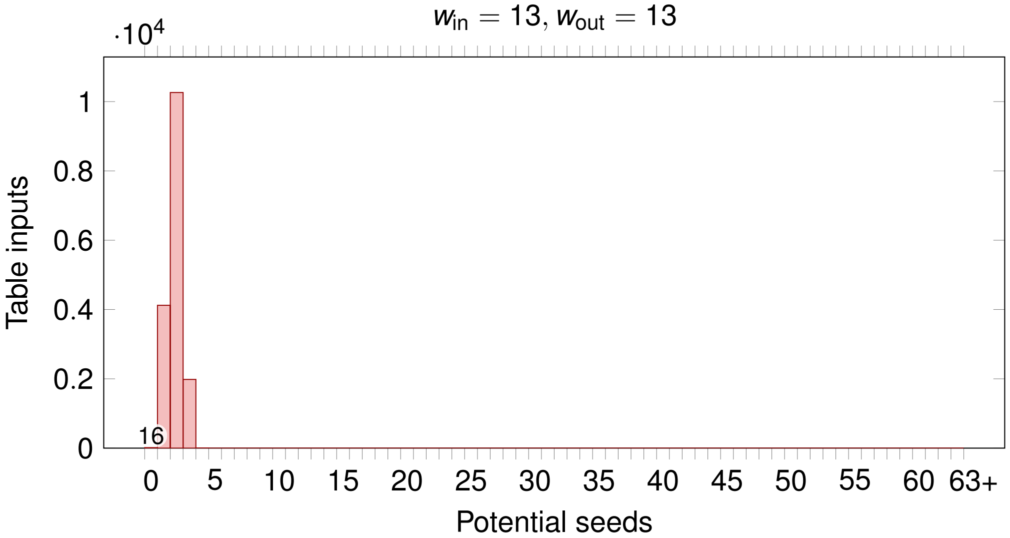

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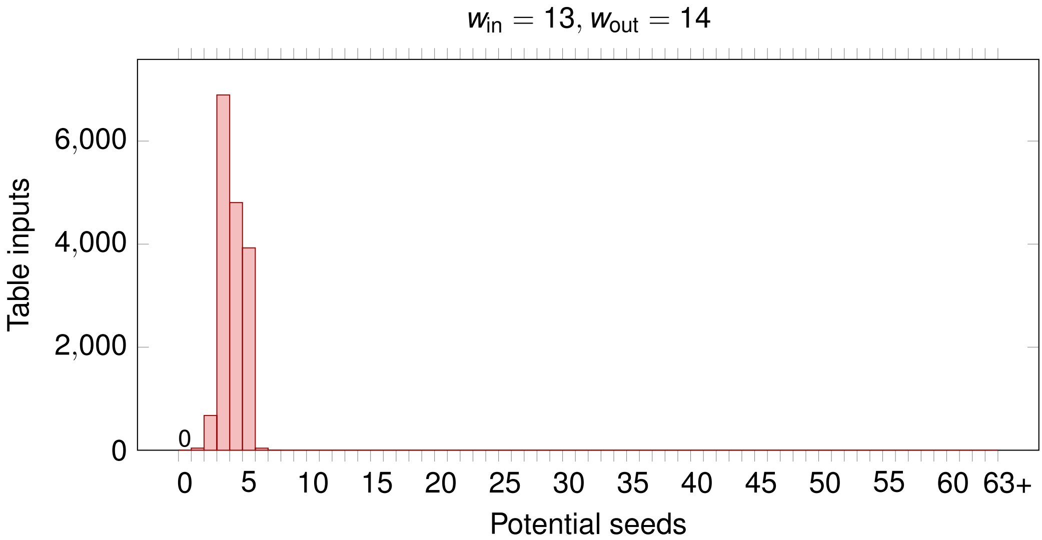

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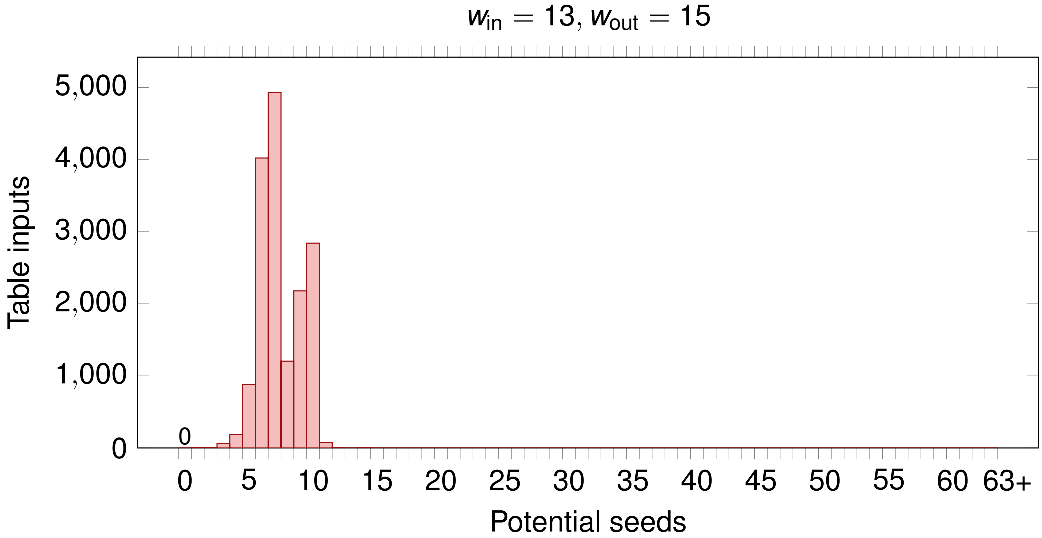

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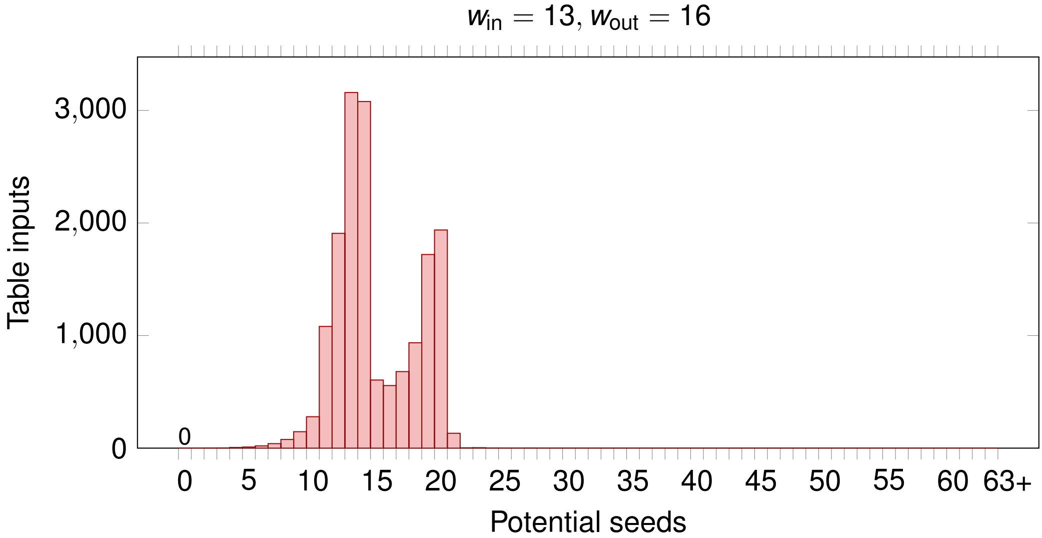

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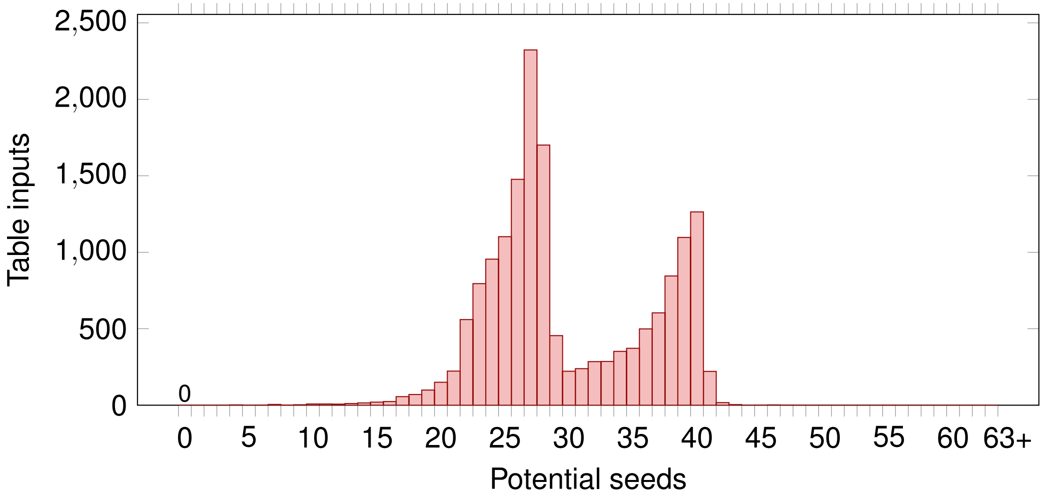
$w_{\text{in}} = 13, w_{\text{out}} = 17$



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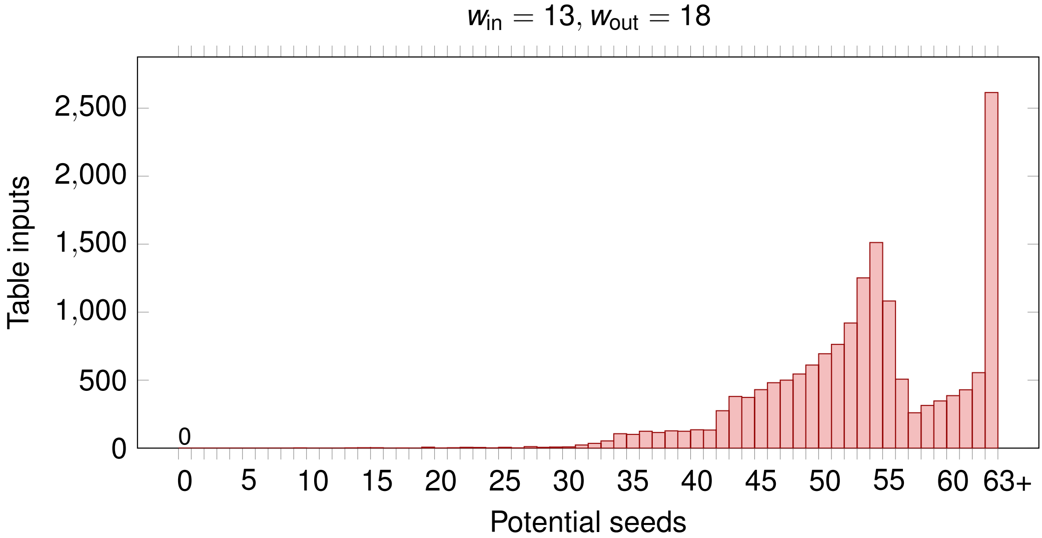

Good news: the table exists!

Table size

$w_{\text{in}} = 13$, $w_{\text{out}} = 14$ is a seed table that stores $2 \times 2^{13} \times 14 = 229\,376$ bit

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There are multiple possible seeds for each input

One combination might compress better than another.

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As suspected, this is quite large and we need a way to compress this table

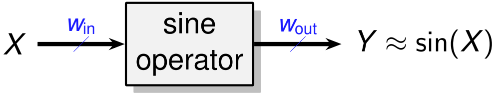
There are multiple possible seeds for each input

One combination might compress better than another.
Send an interval of possible seeds instead of an instance of the table into compression, hoping for a better result.

Multipartite Tables

Goal: build architectures evaluating mathematical functions

Say you have a fixed-point value X, and you need hardware that computes its sine.



The simplest solution: plain tabulation

$2^{w_{\text{in}}}$ entries of w_{out} bits each, so $2^{w_{in}} \times w_{out}$ bits

- very good for really small precisions
- for larger precisions, *cost grows exponentially in* w_{in}

(but only linearly in w_{out})

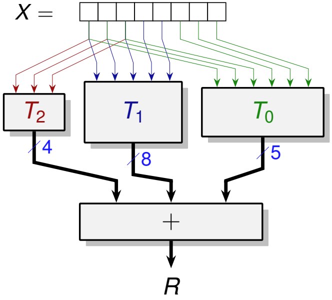
address	content
00000	00000
00001	00011
00010	00011
00011	00101
...	...
11101	11111
11110	11111
11111	11111

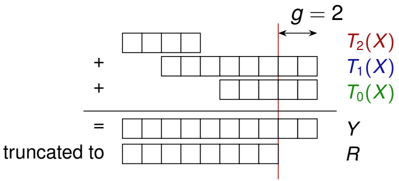
$2^{w_{\text{in}}}$

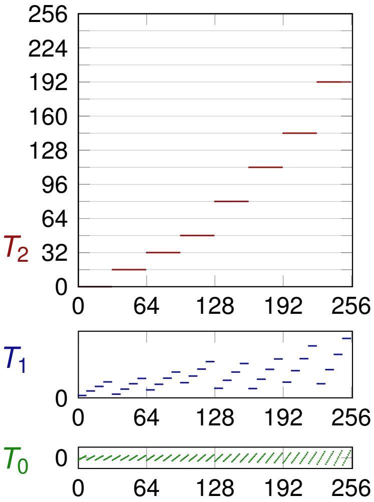
w_{out}

When simple solutions don’t scale, use clever solutions

Multipartite table architectures







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Multipartite table architectures

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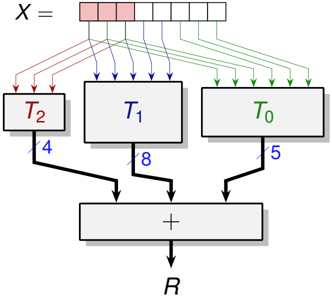
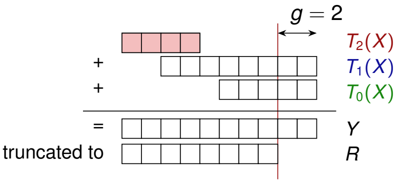
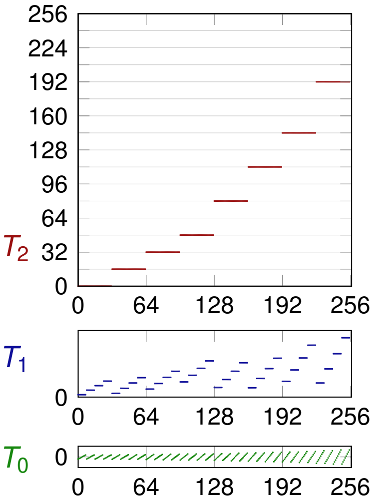
let us just wonder that it works, here for $w_{in} = w_{out} = 8$.







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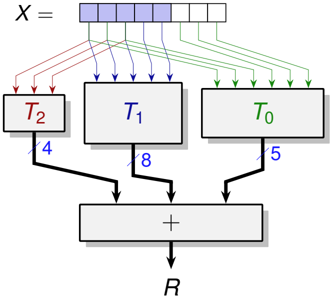

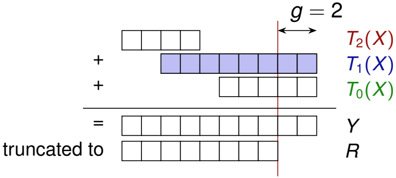

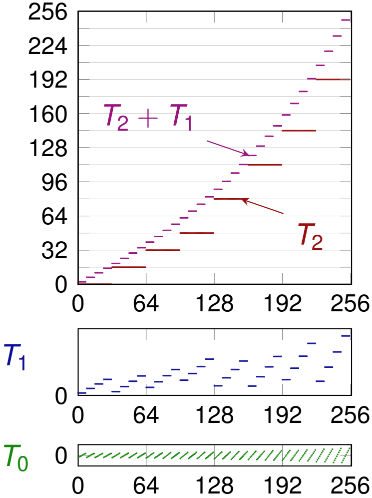

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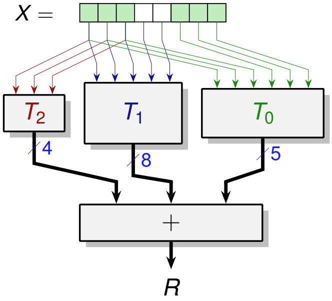

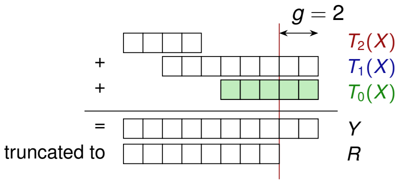

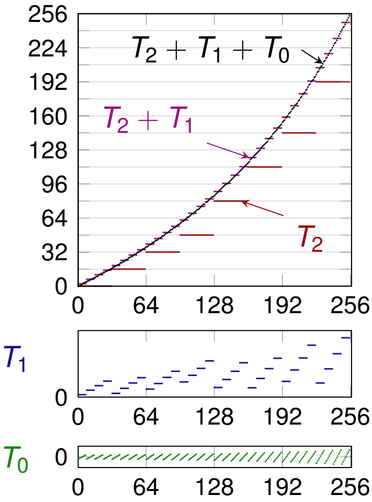

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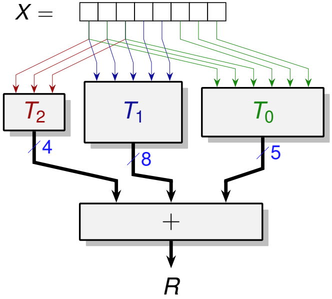

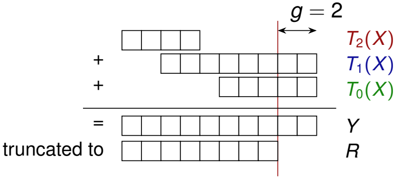

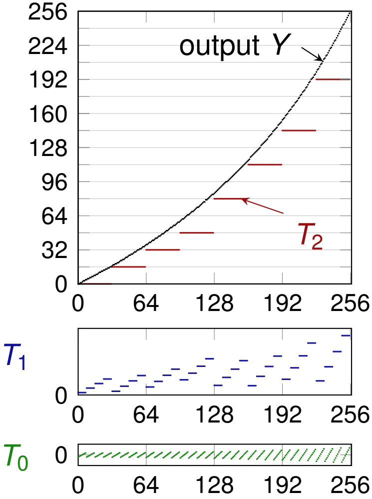

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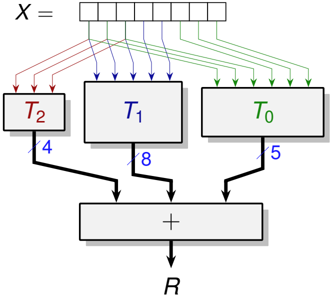
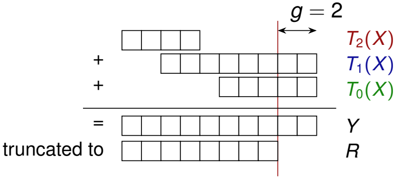
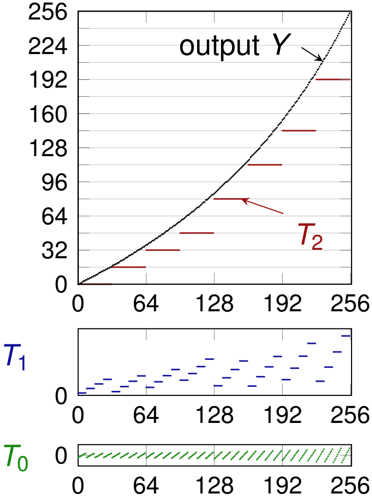


$$2^4 \cdot 4 + 2^5 \cdot 8 + 2^6 \cdot 5 < 2^8 \cdot 8$$





This magic is has a long history spread over 40 years

[1] D. A. Sunderland, R. A. Strauch, S. S. Wharfield, H. T. Peterson, and C. R. Role, “CMOS/SOS frequency synthesizer LSI circuit for spread spectrum communications,” *IEEE Journal of Solid-State Circuits*, 1984.

[2] D. Das Sarma and D. Matula, “Faithful bipartite ROM reciprocal tables,” in *12th Symposium on Computer Arithmetic*, 1995.

[3] H. Hassler and N. Takagi, “Function evaluation by table look-up and addition,” in *12th Symposium on Computer Arithmetic*, 1995.

[4] J. Stine and M. Schulte, “The symmetric table addition method for accurate function approximation,” *Journal of VLSI Signal Processing*, 1999.

[5] J.-M. Muller, “A few results on table-based methods,” *Reliable Computing*, 1999.

[6] F. de Dinechin and A. Tisserand, “Multipartite table methods,” *IEEE Transactions on Computers*, 2005.

[7] S.-F. Hsiao, P.-H. Wu, C.-S. Wen, and P. K. Meher, “Table size reduction methods for faithfully rounded lookup-table-based multiplierless function evaluation,” *Transactions on Circuits and Systems II*, 2015.

[8] M. Christ, L. Forget, and F. de Dinechin, “Lossless differential table compression for hardware function evaluation,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, 2022.

Try me in FloPoCo!

`http://www.flopoco.org`

```
flopoco FixFunctionByMultipartiteTable
tablecompression=true
      f="65535/65536*sin(pi/2*x)" lsbIn=-16 lsbOut=-16
```

Multipartite Tables with Integer Linear Programming (ILP)[1]

Current ILP solvers are extremely powerful, with thousands of variables and constraints.

Express your problem as:

- a set of **variables**,
- a set of **linear constraints** between these variables (e.g. $17x + 42y < 2000$ *and* $y > 0$)
- a **linear cost function** to minimize (e.g. *minimize* $3x - 2y$)

---

[1] O. Desrentes, F. de Dinechin, *Using integer linear programming for correctly rounded multipartite architectures*, 2022 (ICFPT)

The (very) big picture of the ILP

Notation: ILP variables in red, ILP constants in blue.

- Let us have one boolean variable for each bit stored in a table
  - $b_{t,a,i}$ is bit i of the value stored at address a in table T_t

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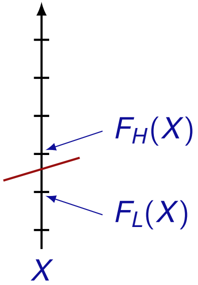
- Let us have 2^n constraints, one for each value of X
  (which then becomes a constant in each of these constraints)

$$\mathcal{C}_X : F_L(X) \leq Y(X) \leq F_H(X)$$



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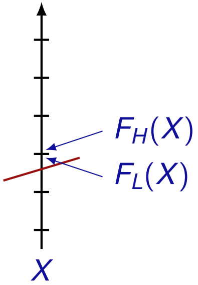
... with $F_L(X) = F_H(X)$ for correct rounding



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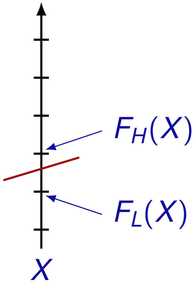

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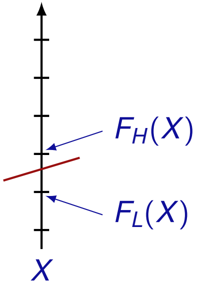

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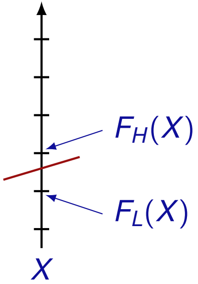

- Now $Y(X)$ can be replaced with $\sum_t T_t(X)$
- ...where the value of $T_t(X)$ is linear in our variables: $\sum_i 2^i b_{t,a,i}$
- so eventually each constraint is indeed linear, something like

$$\mathcal{C}_X : F_L(X) \leq \sum_t \sum_i 2^i b_{t,a(X),i} \leq F_H(X)$$

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Let's plot the table, and see.

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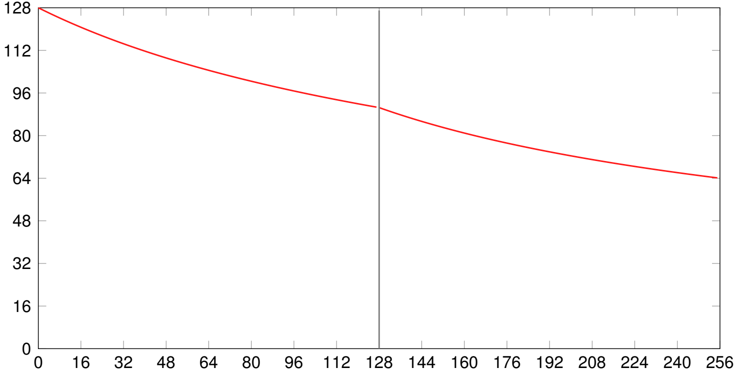

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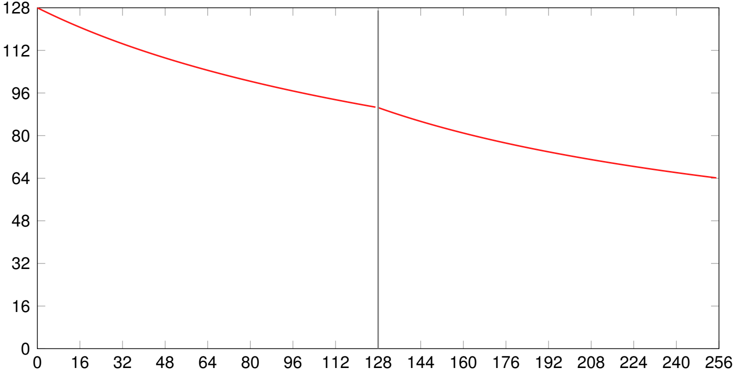

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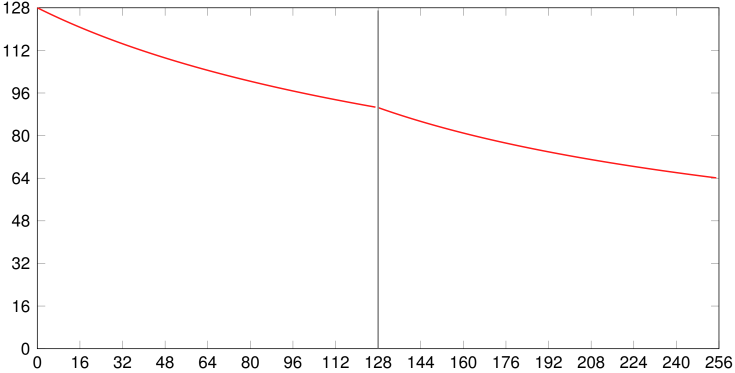

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Seed table is not a function $x \mapsto f(x)$ but $x \mapsto [y_L; y_H]$.

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[1] M. Christ, L. Forget, and F. de Dinechin, *Lossless differential table compression for hardware function evaluation*, 2022 (TCAS-II)

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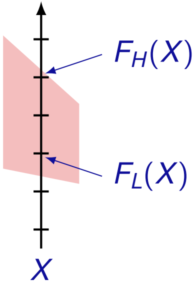
$F_L(X)$ and $F_H(X)$ are exactly what we needed.



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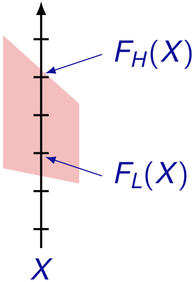

This isn't generic table compression like LDTC[1], and **technically** we're not compressing **a** table, but any of the possible tables.

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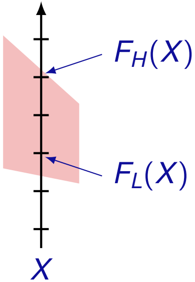

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In particular, we want to compress the one with the smallest compressed size…

---

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Results of the Integer Linear Program

Issues with $w_{\text{in}} = 13, w_{\text{out}} = 14$

The ILP program cannot find cheaper than the plain table. . . Let's try to increase w_{out}.

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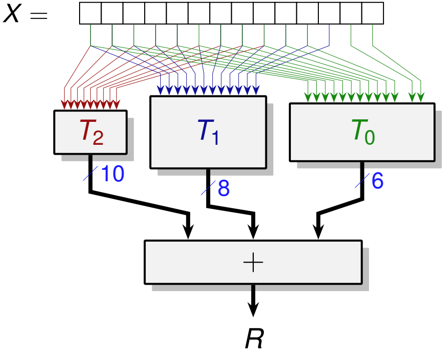
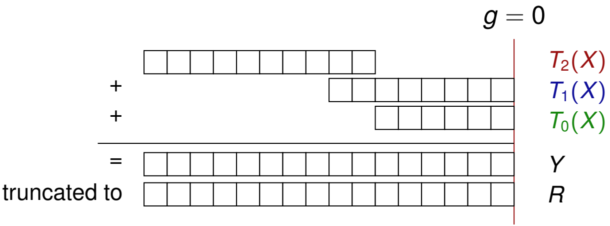
$w_{\text{out}} = 15$



$$2^{9} \cdot 10 + 2^{12} \cdot 8 + 2^{12} \cdot 6 < 2^{14} \cdot 15$$



Results of the Integer Linear Program

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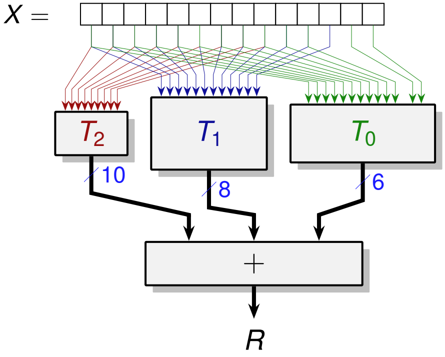
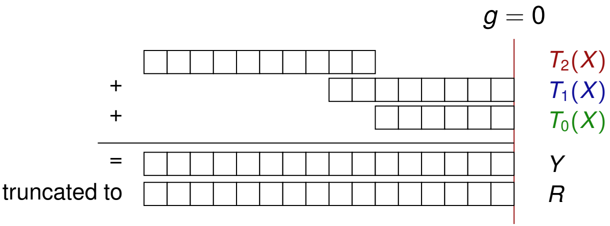
$w_{\text{out}} = 15$; **25.4%** of initial table size



$$2^9 \cdot 10 + 2^{12} \cdot 8 + 2^{12} \cdot 6 < 2^{14} \cdot 14$$



Results of the Integer Linear Program

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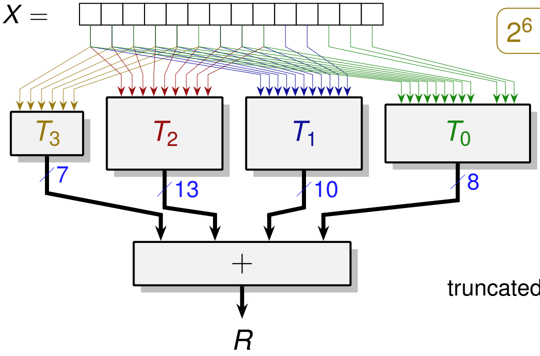
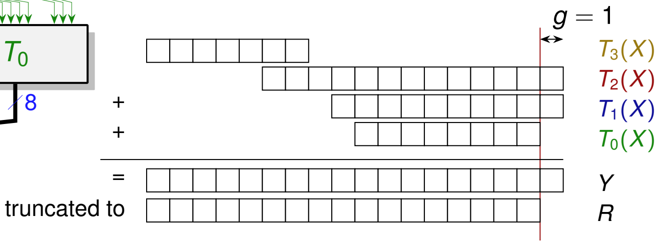
The ILP program cannot find cheaper than the plain table... Let's try to increase w_{out}.

$w_{\text{out}} = 16$



$$2^6 \cdot 7 + 2^9 \cdot 13 + 2^{10} \cdot 10 + 2^{11} \cdot 8 < 2^{14} \cdot 16$$

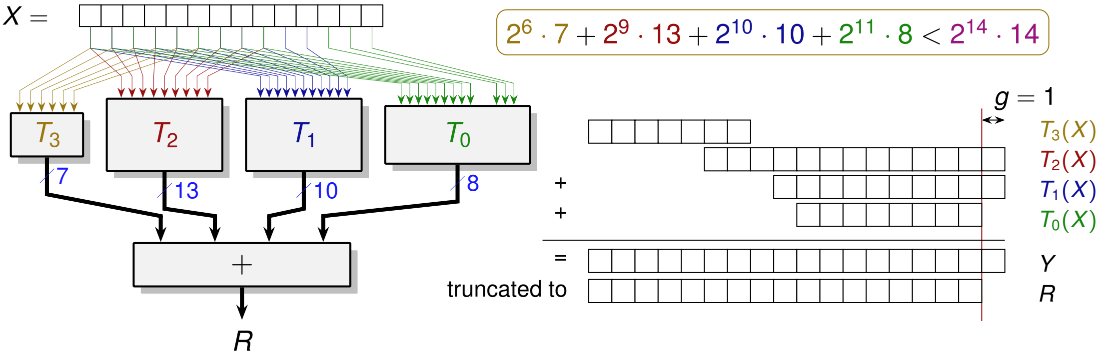


Results of the Integer Linear Program

Issues with $w_{\text{in}} = 13$, $w_{\text{out}} = 14$

The ILP program cannot find cheaper than the plain table… Let's try to increase w_{out}.

$w_{\text{out}} = 16$; **13.7%** of initial table size



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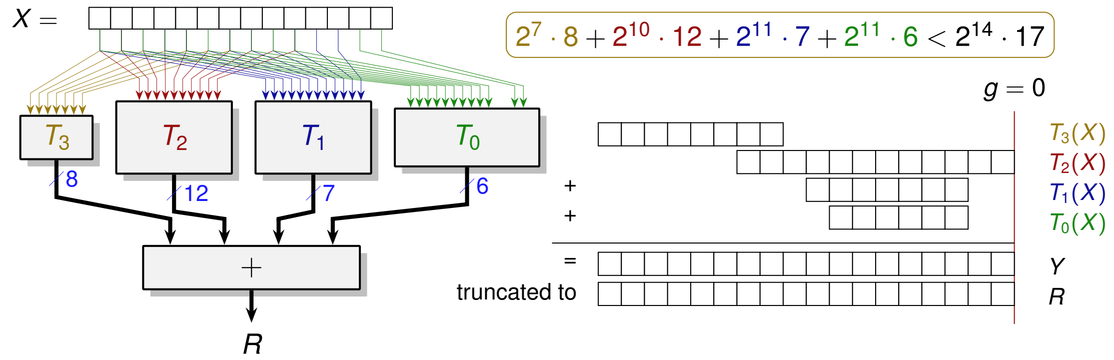

$$2^7 \cdot 8 + 2^{10} \cdot 12 + 2^{11} \cdot 7 + 2^{11} \cdot 6 < 2^{14} \cdot 17$$

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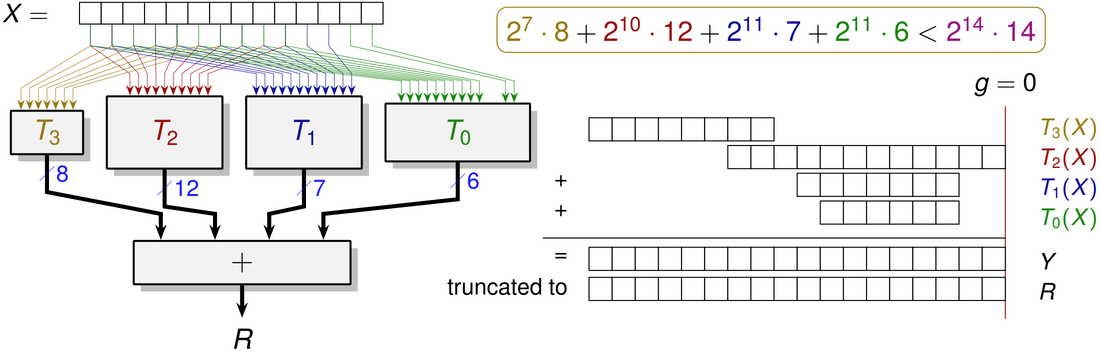
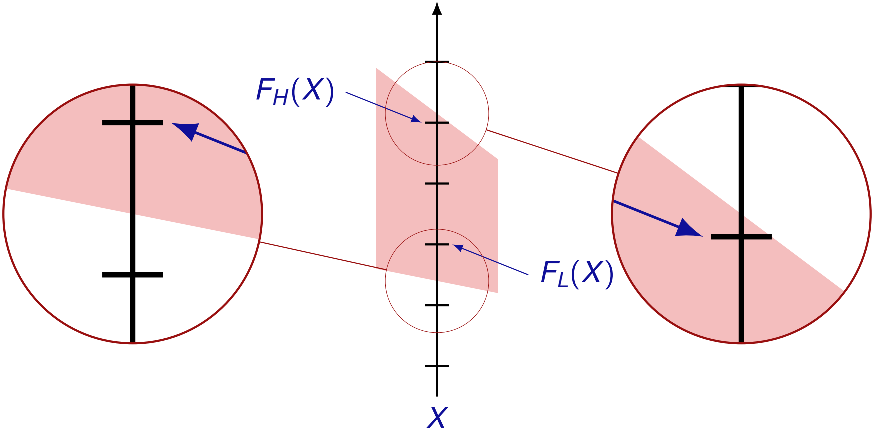


$$2^7 \cdot 8 + 2^{10} \cdot 12 + 2^{11} \cdot 7 + 2^{11} \cdot 6 < 2^{14} \cdot 14$$

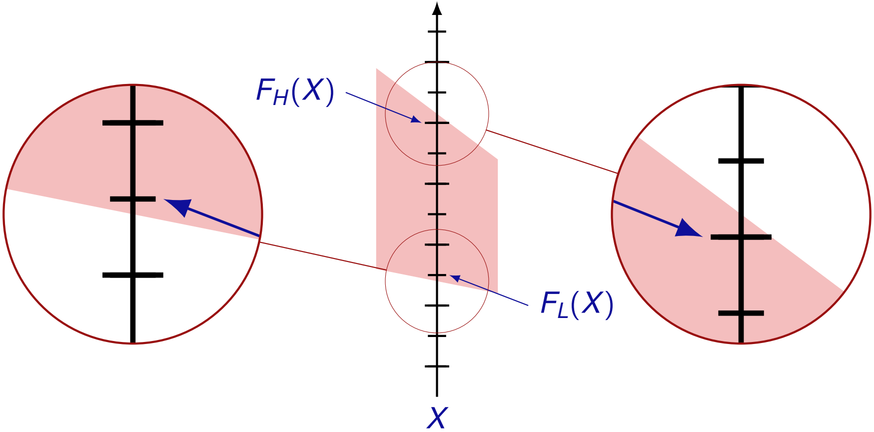
Why is that?

When $w_{\text{out}} = 14$: 3 seeds, interval size 2×2^{-14}



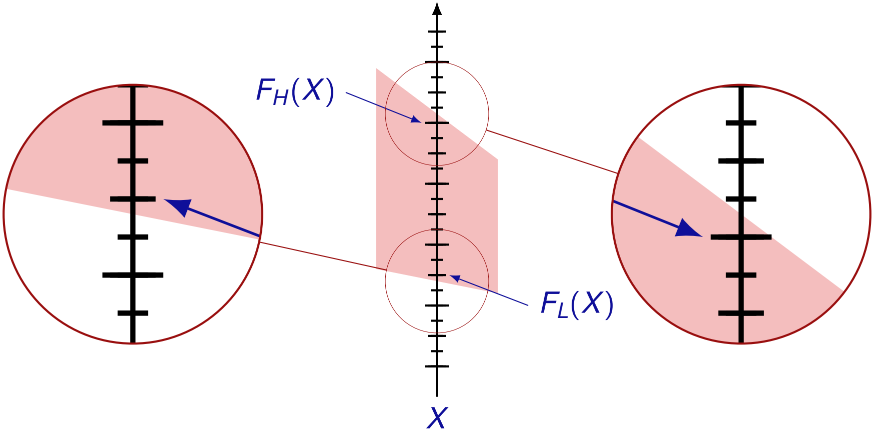
Why is that?

When $w_{\text{out}} = 15$: 6 seeds, interval size 2.5×2^{-14}



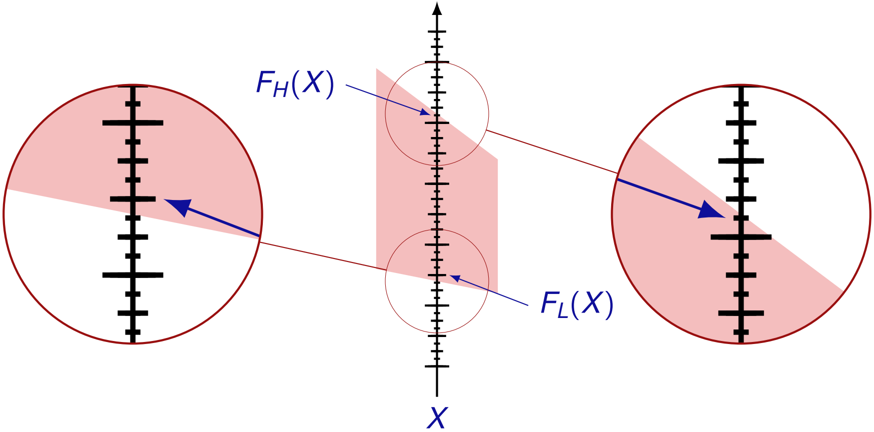
Why is that?

When $w_{\text{out}} = 16$: 11 seeds, interval size 2.5×2^{-14}



Why is that?

When $w_{\text{out}} = 17$: 22 seeds, interval size 2.625×2^{-14}



Why is that?

→ But why does it get bigger again ?

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The interval is a simplification. Example: the seed is `0x3f82fc00`

Why is that?

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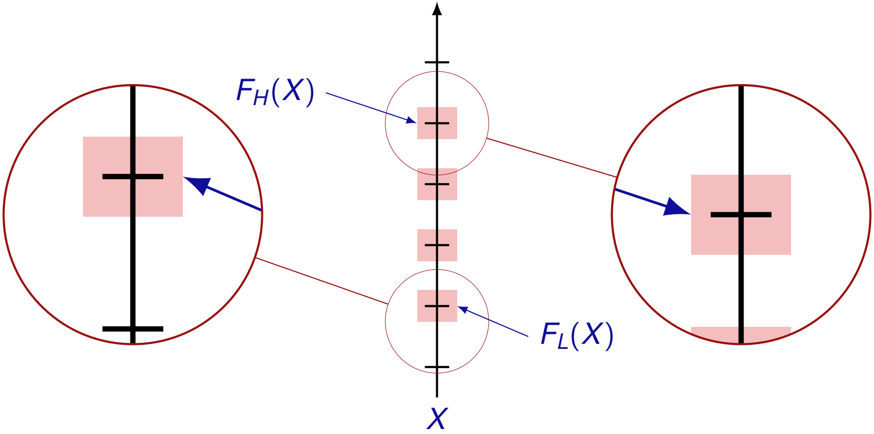
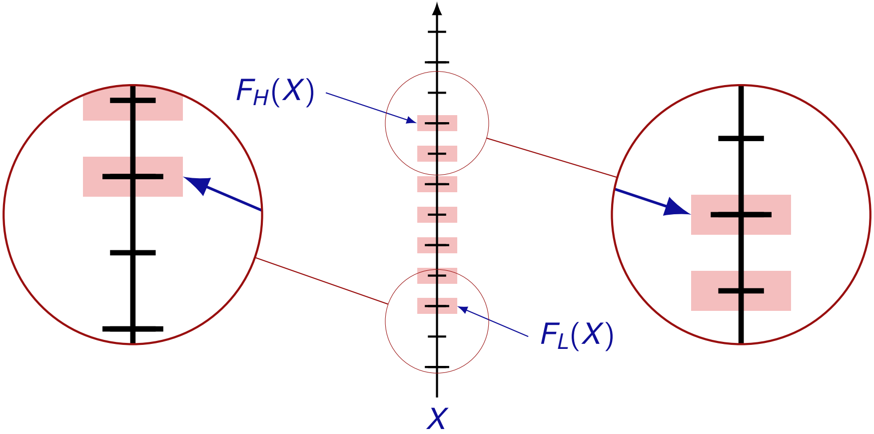
When $w_{\text{out}} = 14$: 4 seeds, interval size 3×2^{-14}



Why is that?

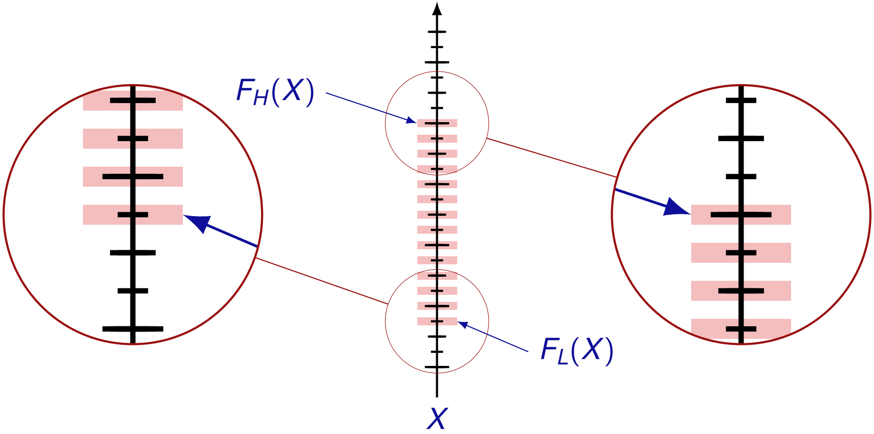
The interval is a simplification. Example: the seed is `0x3f82fc00`
When $w_{\text{out}} = 15$: 7 seeds, interval size 3×2^{-14}



Why is that?

The interval is a simplification. Example: the seed is `0x3f82fc00`

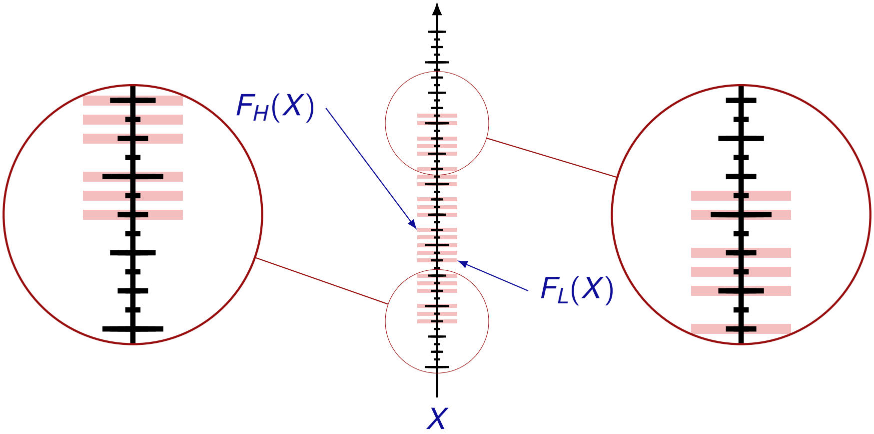
When $w_{\text{out}} = 16$: 14 seeds, interval size 3.25×2^{-14}



Why is that?

The interval is a simplification. Example: the seed is `0x3f82fc00`

When $w_{\text{out}} = 17$: 5 seeds, interval size 0.5×2^{-14}



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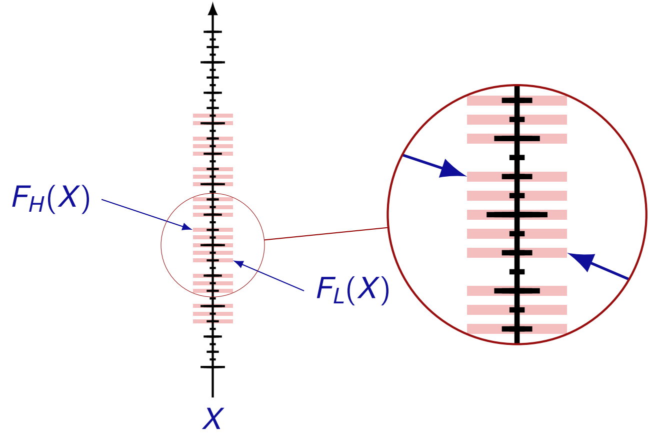

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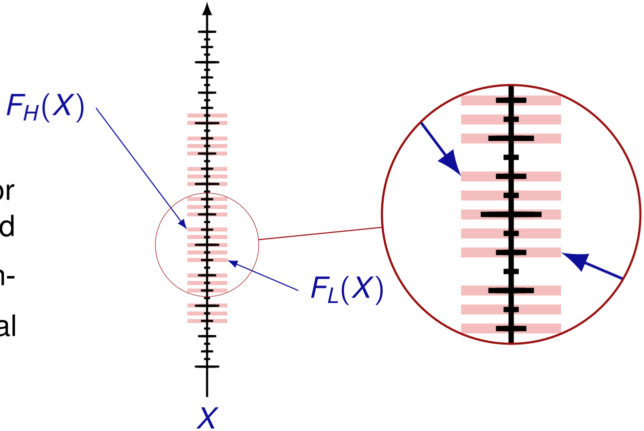

The algorithm for faithfully rounded FP32 $\frac{1}{\sqrt{x}}$ is sensitive to the initial seed.

Conclusion

What we wanted

Cheap vector normalisation for ML (and hopefully more)

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The initial plan

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The initial plan

- Phase 1: Find the perfect seed table

Conclusion

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- Phase 1: Find the perfect seed table
- Phase 2: Stick it in an ILP solver

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What we found along the way

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- Phase 1: Find the perfect seed table
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What we found along the way

- The perfect table does not exist.

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- Increasing w_{out} reduces the compressed size.

Conclusion

What we wanted

Cheap vector normalisation for ML (and hopefully more)

The initial plan

- Phase 1: Find the perfect seed table
- Phase 2: Stick it in an ILP solver
- Phase 3: Profit

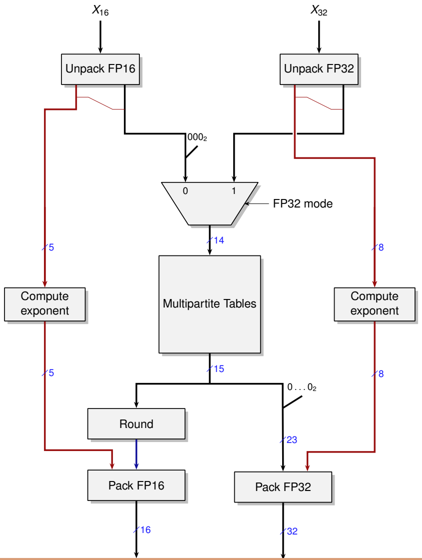
What we found along the way

- The perfect table does not exist. That's actually better for the ILP.
- Guard bits for table compression do not work like the ones for function evaluation. Similar to LDTC, no guard bits
- Increasing w_{out} reduces the compressed size. Until we choose the wrong seed representation. . .

Next steps

Next steps

- Actually build the hardware operator



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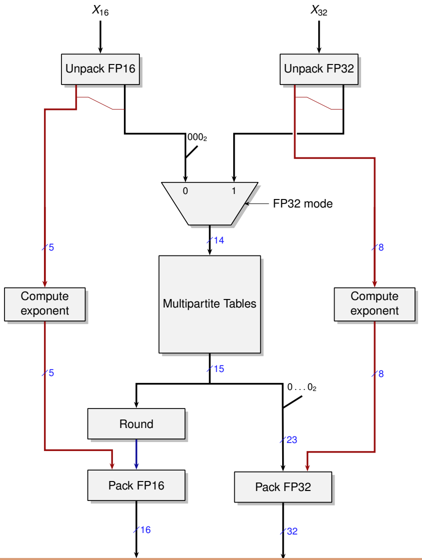
- Actually build the hardware operator
- Set up the software programs to use it



Next steps

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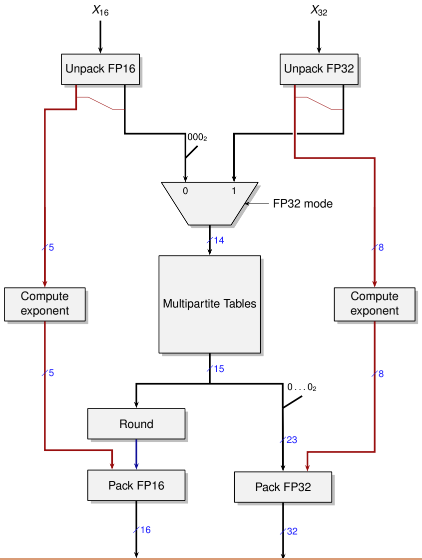

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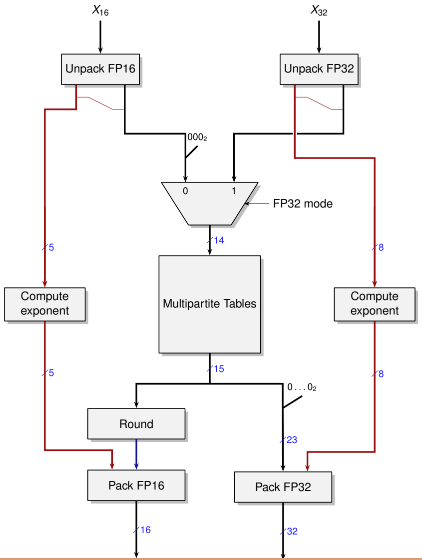
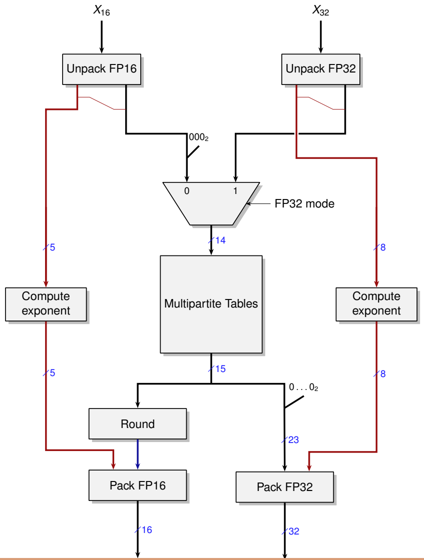
- Check we can use the seed for division (brute-force will not work anymore. . .)



Next steps

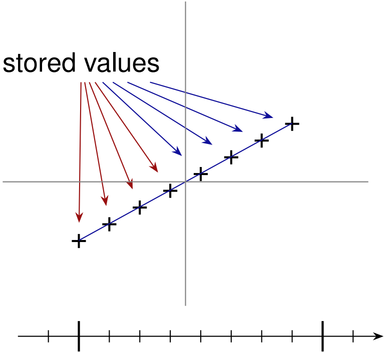
- Actually build the hardware operator
- Set up the software programs to use it
- Reflect on the limitations of the interval representation
- Check we can use the seed for division (brute-force will not work anymore. . .)
- Try to compress other tables using multipartite architectures

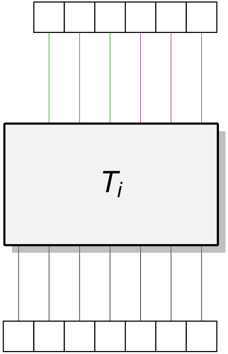


Questions?

Outline

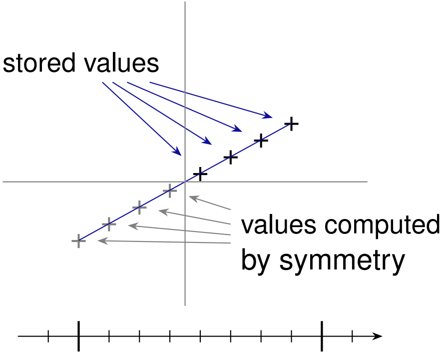
To use or not to use symmetry





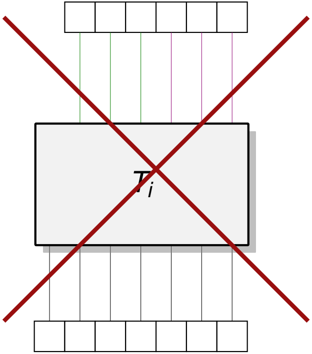
To use or not to use symmetry

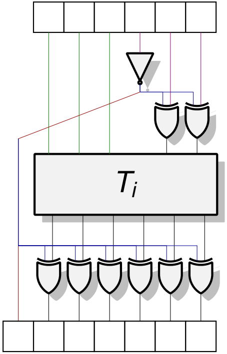
Symmetry may trade one table input bit for two rows of XOR gates:



stored values

values computed by symmetry





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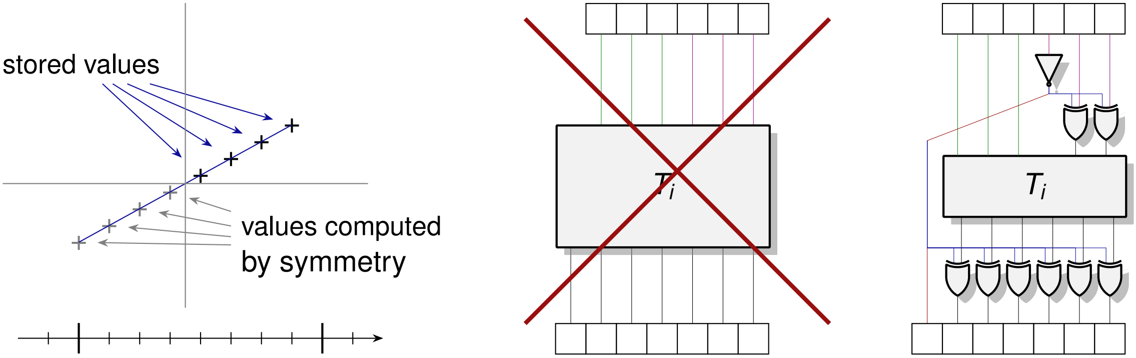

- The XORs save LUTs, but cost LUTs:
  - detailed evaluation of the relevance of this idea in the paper

Results of the Integer Linear Program

Issues with $w_{\text{in}} = 13$, $w_{\text{out}} = 14$

The ILP program cannot find cheaper than the plain table. . . Let's try to increase w_{out}.

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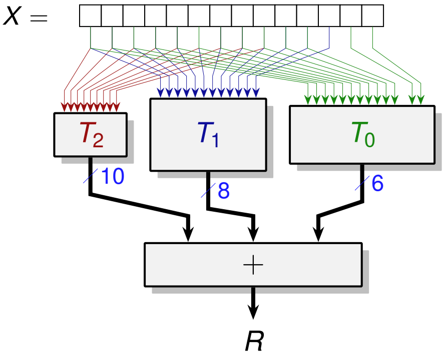
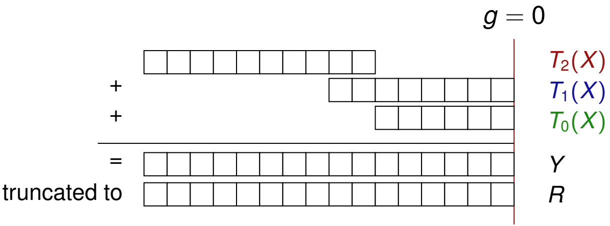
$w_{\text{out}} = 15$



$$2^9 \cdot 10 + 2^{12} \cdot 8 + 2^{12} \cdot 6 < 2^{14} \cdot 15$$



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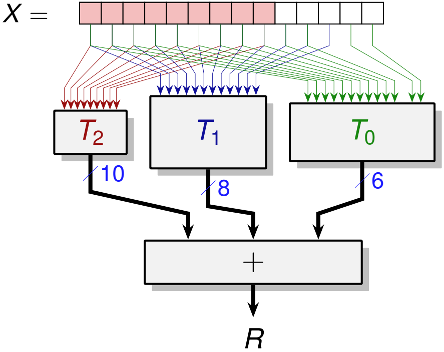

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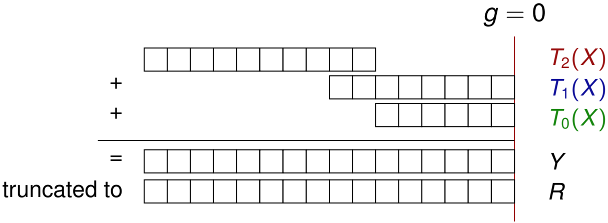

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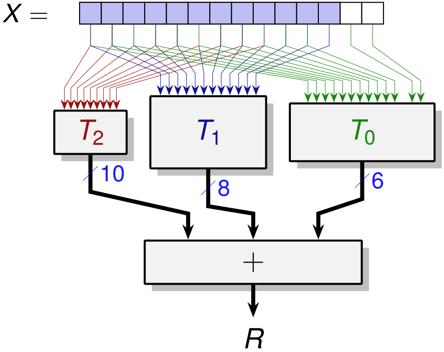

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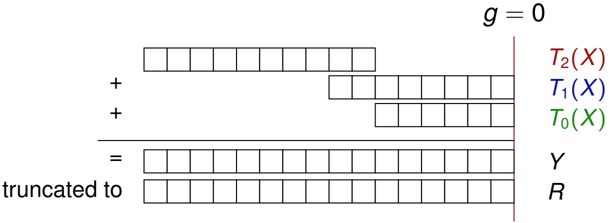

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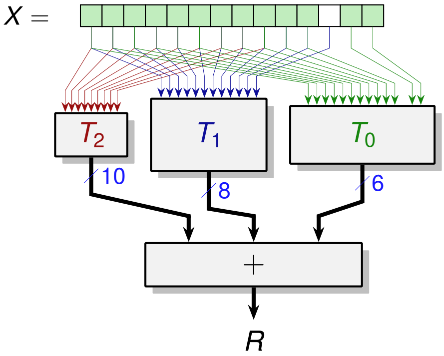

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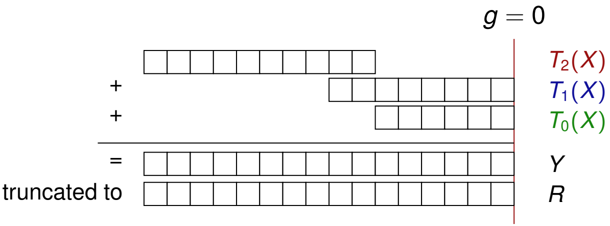
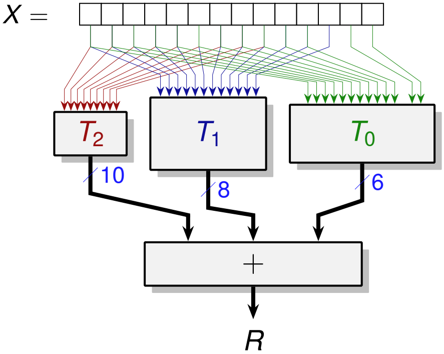


Results of the Integer Linear Program

Issues with $w_{\text{in}} = 13$, $w_{\text{out}} = 14$

The ILP program cannot find cheaper than the plain table... Let's try to increase w_{out}.

$w_{\text{out}} = 15$; **25.4%** of initial table size



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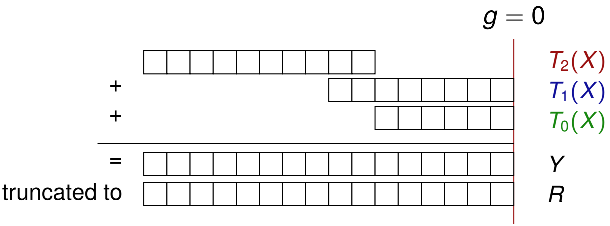

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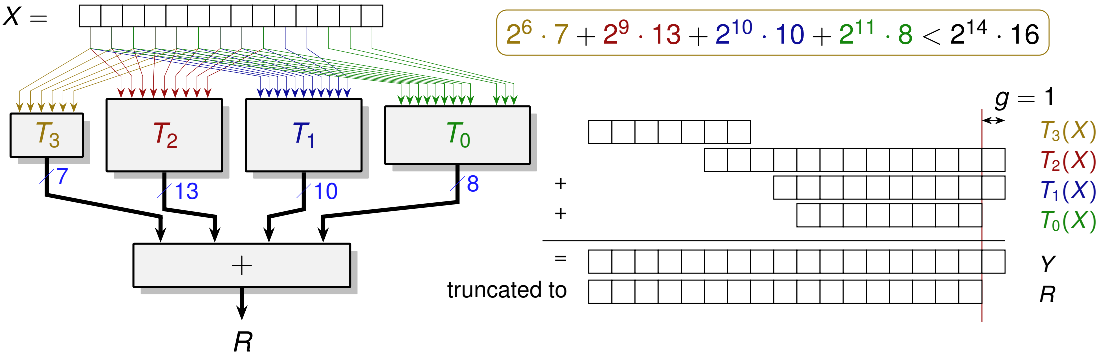
$w_{\text{out}} = 16$



$$2^6 \cdot 7 + 2^9 \cdot 13 + 2^{10} \cdot 10 + 2^{11} \cdot 8 < 2^{14} \cdot 16$$

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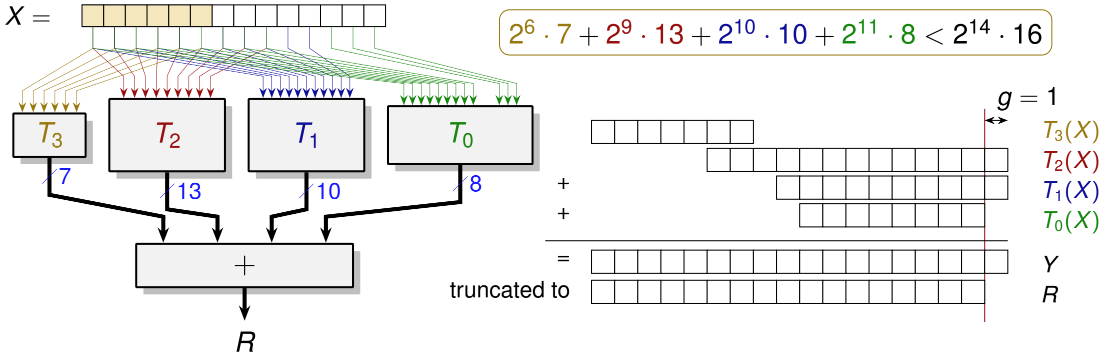

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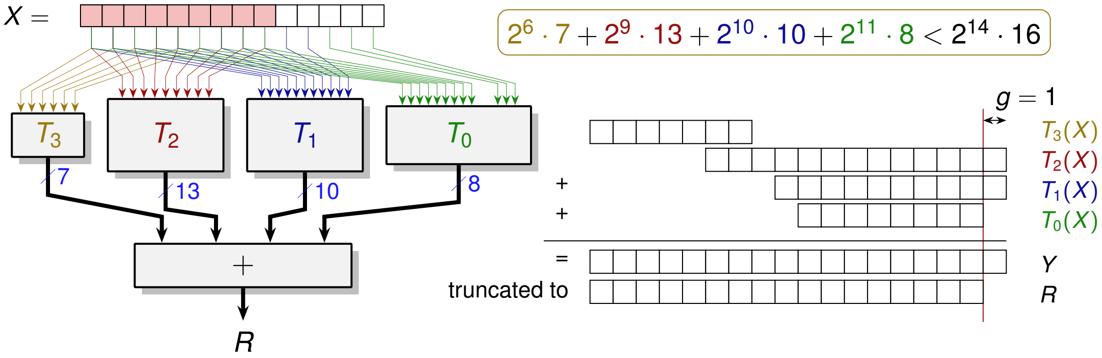

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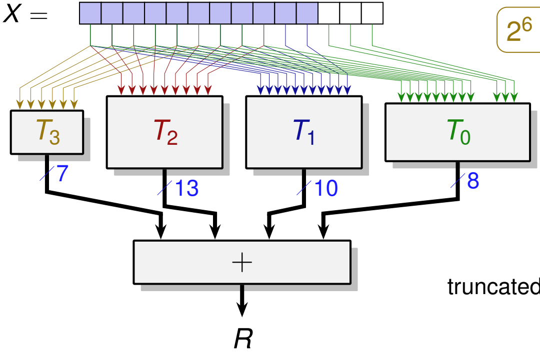

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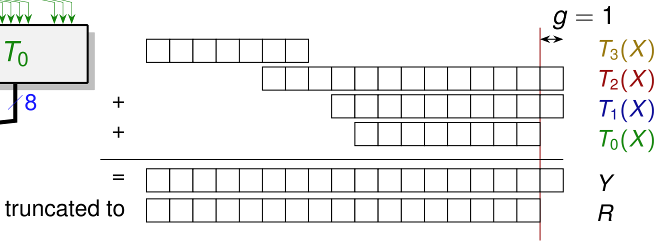

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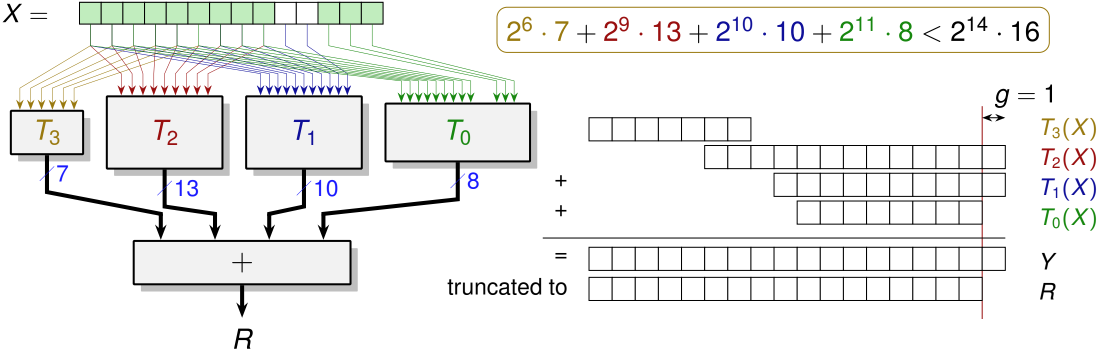

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Results of the Integer Linear Program

Issues with $w_{\text{in}} = 13$, $w_{\text{out}} = 14$

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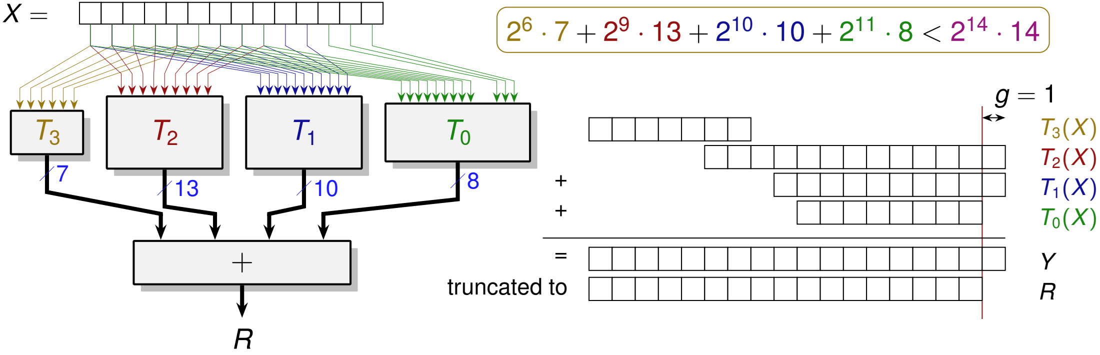
$w_{\text{out}} = 16$; **13.7%** of initial table size



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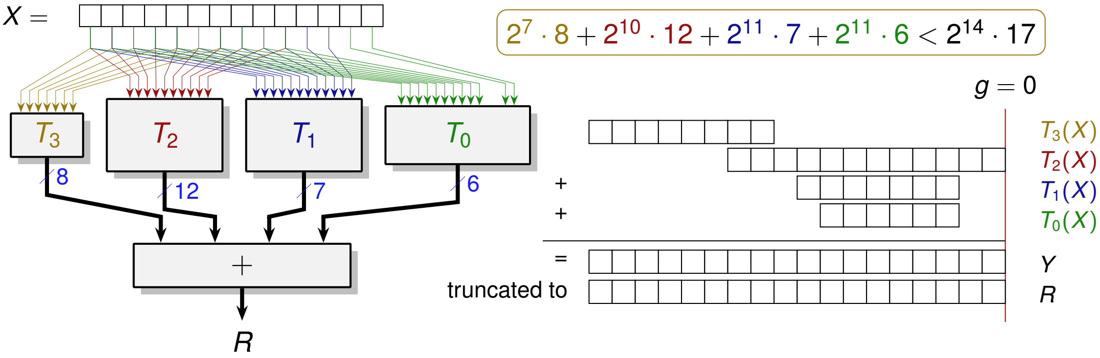

$$2^7 \cdot 8 + 2^{10} \cdot 12 + 2^{11} \cdot 7 + 2^{11} \cdot 6 < 2^{14} \cdot 17$$

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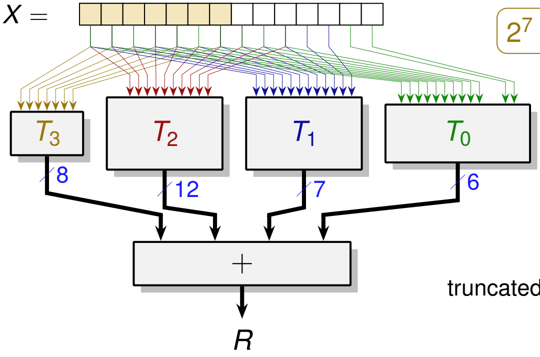

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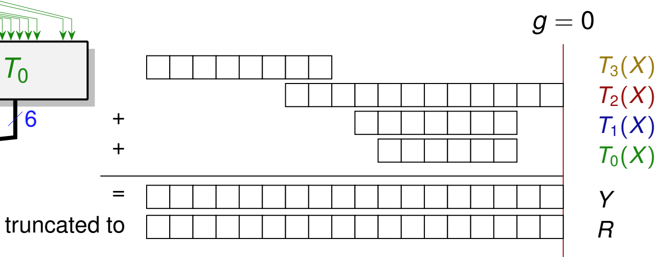

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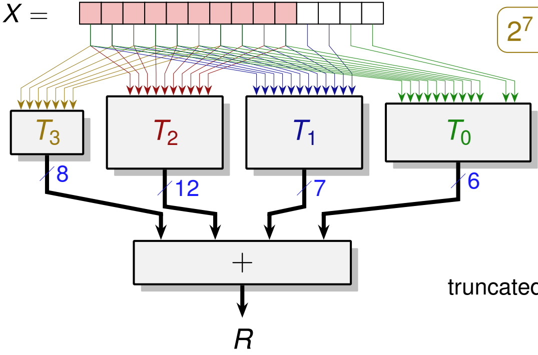

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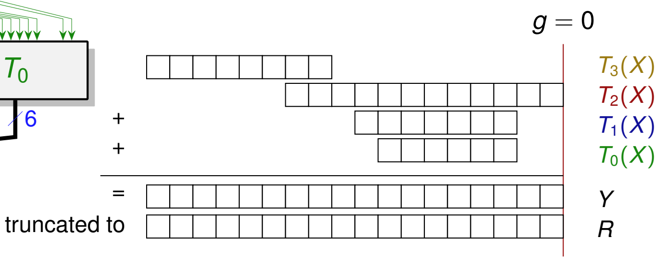

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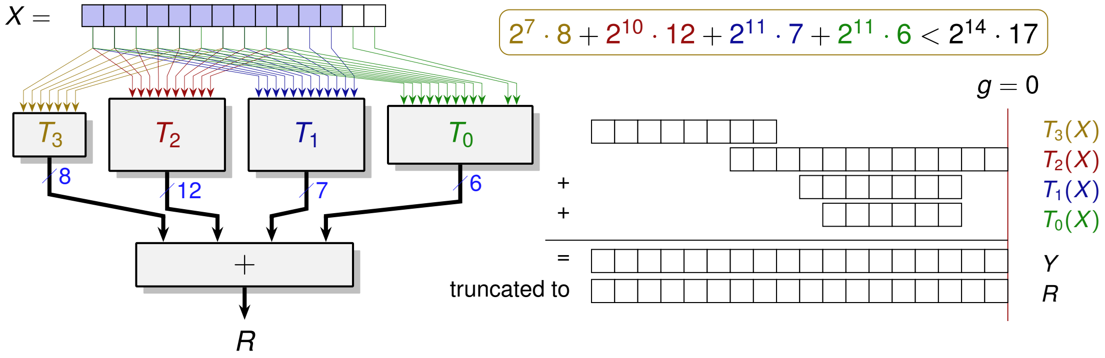

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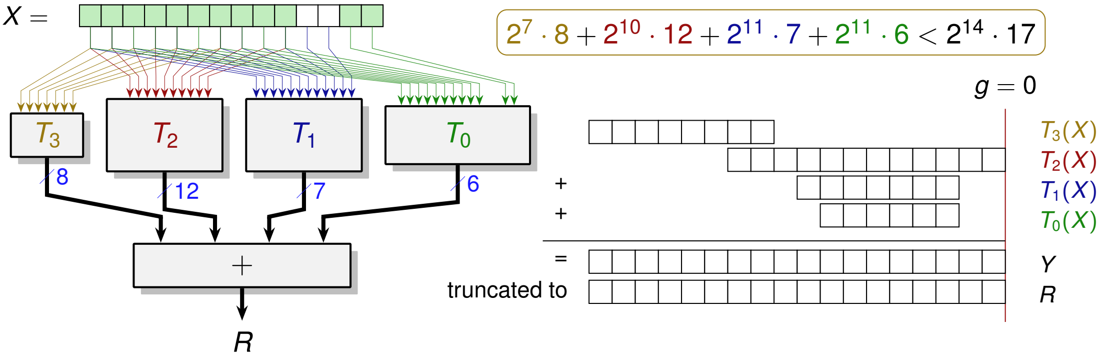

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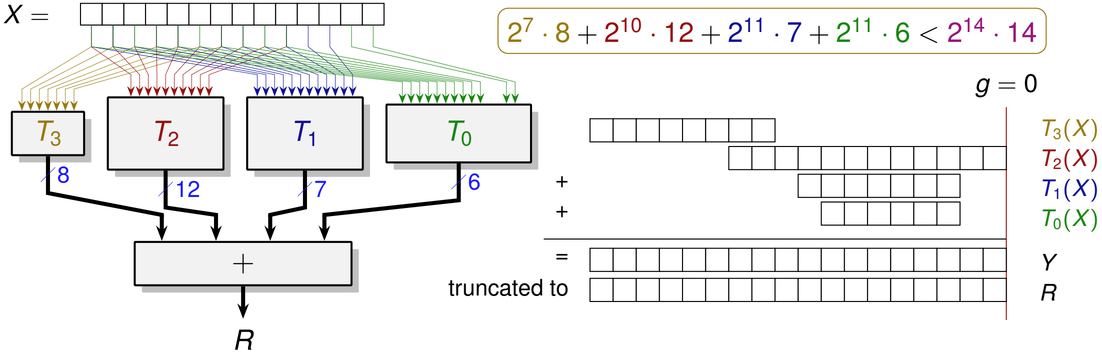
The ILP program cannot find cheaper than the plain table. . . Let's try to increase w_{out}.

$w_{\text{out}} = 17$; **17.4%** of initial table size



$$2^7 \cdot 8 + 2^{10} \cdot 12 + 2^{11} \cdot 7 + 2^{11} \cdot 6 < 2^{14} \cdot 14$$

Speed and scaling of the algorithms

Increasing w_{out} does not make the program exponentially slower.

Tested for $w_{\text{out}} \in [\![14, 18]\!]$ on a laptop (intel i7 from 2019, 16 GB RAM)

- Generating the intervals : always less than a minute
- ILP : always less than 30 min

Optimality ?

In short : no

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Many heuristics that potentially removes optimiality.

- Replace some binary variables with integers
- Prune the external loop
- Set a maximum amount of guard bits
- **ILP timeout**

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Optimality ?

In short : no

Many heuristics that potentially removes optimiality.

- Replace some binary variables with integers
- Prune the external loop ← This also could have an impact
- Set a maximum amount of guard bits
- **ILP timeout** ← I suspect this is the worse offender for larger tables

Negative numbers in $\frac{1}{x}$?

Hardware operator raises a flag when input is negative, and outputs $\frac{1}{\sqrt{|x|}}$.

FMA that computes the square of the seed will compute $0 - a \times b$ instead of $0 + a \times b$ if the flag is raised.

Subnormals

In hardware

No subnormal output possible (as $\frac{1}{\sqrt{x}}$ reduces the exponent)
Could renormalise the subnormal input (costs a shifter, would probably add a cycle). I think it is worth it, but the µm^2 is expensive.

In software

Will create problems for $\frac{1}{x}$:

- Iterations are not done with range reduction, can create issues.
- Double rounding can happen on subnormal outputs[1]. Not sure if the solution will still work in my case

---

[1] JM Muller, *Avoiding double roundings in scaled Newton-Raphson division*, ASILOMAR (2013)

$\frac{1}{x}$ and the last seed before a power of 2

The input $F = $`0b111...111` was removed from the seed table generation program, as the software iterations do not result in a correctly rounded $\frac{1}{x}$.

---

[1] M. Cornea-Hasegan, R. Golliver, P. Markstein, *Correctness proofs outline for Newton-Raphson based floating-point divide and square root algorithm*, 1999 (ARITH)

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Previously studied [1] as a tricky case.

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The input $F =$`0b111...111` was removed from the seed table generation program, as the software iterations do not result in a correctly rounded $\frac{1}{x}$.
Previously studied [1] as a tricky case.
The fix is probably going to be a conditional move in software when correct rounding is needed.

---

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FP16 $\frac{1}{\sqrt{a}}$ CR

```
$o \Leftarrow$ seed($a$) (c0)
```

Latency : 1 cycle
Throughput : 16 op/cycle

FP32 $\frac{1}{\sqrt{a}}$ FR

```
$y_0 \Leftarrow \text{seed}(a)$ (c0)
$h_0 \Leftarrow 0.5 \times y_0$ (c1)
$g_0 \Leftarrow a \times y_0$ (c2)
$r_0 \Leftarrow -g_0 \times h_0 + 0.5$ (c6)
$o \Leftarrow r_0 \times y_0 + y_0$ (c10)
```

Latency : 14 cycles
Throughput : 4 op/cycle

FP16 $\sqrt{a}$ err < 1.4ulp[1]

$y_0 \Leftarrow \text{seed}(a)$ (c0)
$o \Leftarrow a \times y_0$ (c1)

Latency : 5 cycles
Throughput : 16 op/cycle

---

[1] $1.4 \approx \sqrt{2}$: Coincidence ? Probably

FP16 $\sqrt{a}$ CR

$y_0 \Leftarrow \text{seed}(a)$ (c0)
$g_0 \Leftarrow a \times y_0$ (c1)
$h_0 \Leftarrow 0.5 \times y_0$ (c2)
$e_0 \Leftarrow -g_0 \times g_0 + a$ (c5)
$o \Leftarrow e_0 \times h_0 + g_0$ (c9)

Latency : 13 cycles
Throughput : 4 op/cycle

---

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$g_1 \Leftarrow r_0 \times g_0 + g_0$ (c10)
$h_1 \Leftarrow r_0 \times h_0 + h_0$ (c11)
$e_1 \Leftarrow -g_1 \times g_1 + a$ (c14)
$o \Leftarrow e_1 \times h_1 + g_1$ (c18)

Latency : 22 cycles
Throughput : 2.3 op/cycle

FP16 $\frac{1}{a}$ err < 1.4ulp

$y_0 \Leftarrow \text{seed}(a)$ (c0)
$o \Leftarrow y_0 \times y_0$ (c1)

Latency : 5 cycles
Throughput : 16 op/cycle

FP16 $\frac{1}{a}$ CR

$y_0 \Leftarrow \text{seed}(a)$ (c0)
$g_0 \Leftarrow y_0 \times y_0$ (c1)
$e_0 \Leftarrow -g_0 \times a + 1$ (c5)
$o \Leftarrow e_0 \times g_0 + g_0$ (c9)

Latency : 13 cycles
Throughput : 5.3 op/cycle

FP32 $\frac{1}{a}$ FR

$$
\begin{aligned}
&y_0 \Leftarrow \mathrm{seed}(a)\ (\mathrm{c0})\\
&x_0 \Leftarrow y_0 \times y_0\ (\mathrm{c1})\\
&r_0 \Leftarrow -a \times x_0 + 1\ (\mathrm{c5})\\
&o \Leftarrow r_0 \times x_0 + x_0\ (\mathrm{c9})
\end{aligned}
$$

Latency : 13 cycles
Throughput : 5.3 op/cycle

FP32 $\frac{1}{a}$ CR

$y_0 \Leftarrow \mathrm{seed}(a)$ (c0)
$x_0 \Leftarrow y_0 \times y_0$ (c1)
$r_0 \Leftarrow -a \times x_0 + 1$ (c5)
$x_1 \Leftarrow r_0 \times x_0 + x_0$ (c9)
$r_1 \Leftarrow -a \times x_1 + 1$ (c13)
$o \Leftarrow r_1 \times x_1 + x_1$ (c17)

Latency : 21 cycles
Throughput : 3.2 op/cycle