Rounding Errors Statistics as Numerics Signature

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RAIM 2025

A Tribute to Jean-Michel Muller

Lyon, France

Thank you
Jean-Michel
for all your
contributions and
leadership

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Rounding Errors Stats as Numerics Signature

- Motivation
- Proposition
- Validation
- Application

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Motivation: Numerics full of Idiosyncrasies

- Pre-IEEE, different radices, strange properties
- IEEE 754 made a great positive impact, not a panacea though
- We are now in the golden age of computer architecture, ALAS
  - Many specific operator-level accelaration with special numerics

Motivation: Two Specific Needs

- For hardware designers
  - How do we know design spec is the same as a C-model
  - If the C-model IS is the spec, is the numerics what we expect
- For hardware/sofware co-designers
  - Is vendor delivering what we expect, want some practical sanity checks
  - Need some comparisons of new numerics to old

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  - Need some comparisons of new numerics to old
- A standard practice of threshold on a single error quite inadequate

Threshold Test/Single Error – An Example

```
def sgl_simd_sum(x_data, simd_len):
    L = len(x_data); s = 0.0; ind = 0;
    n_outer = L // simd_len
    for _ in range(n_outer):
        tmp = 0.0
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            tmp = tmp + x_data[ind]
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Threshold Test/Single Error – An Example

Expect kernel is **simd_len == 4**
Generate $L = 64$ random inputs
compute error E (vs. double)
Expect $|E| < 18\epsilon$

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Fails because $E = -18.7\epsilon$
Change theshold to 19ϵ
ALL IS WELL

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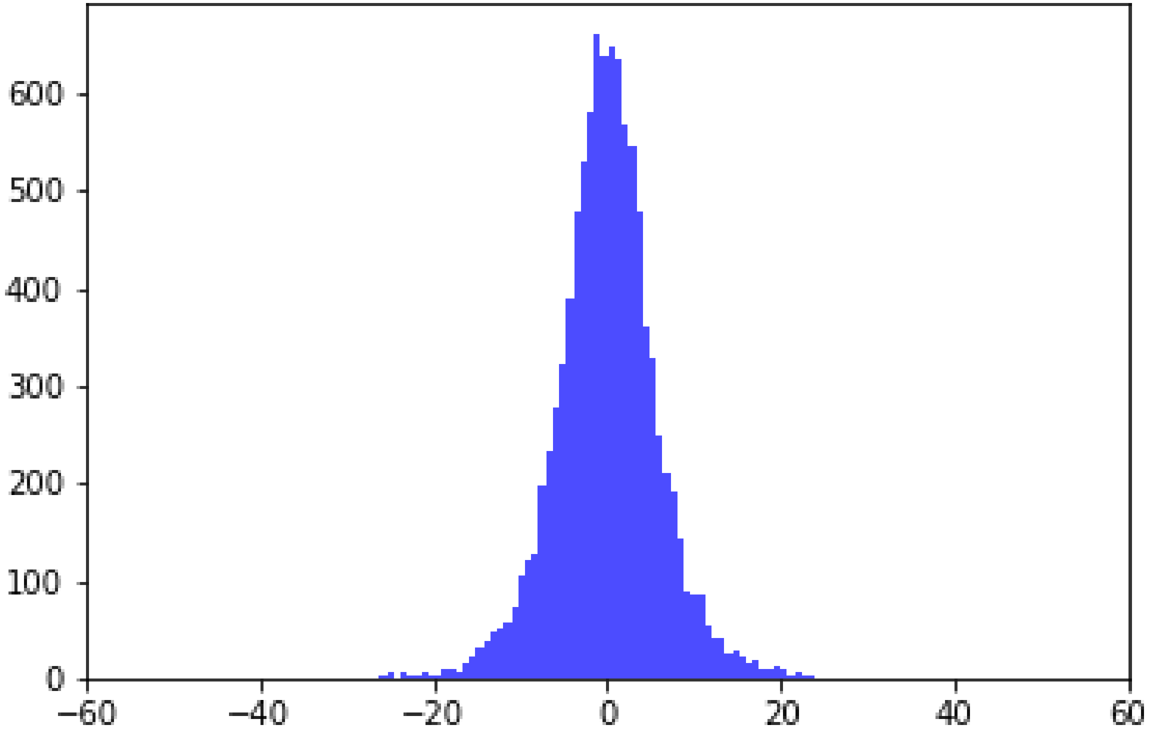

histogram of 10^6 error (units ϵ) simd 4

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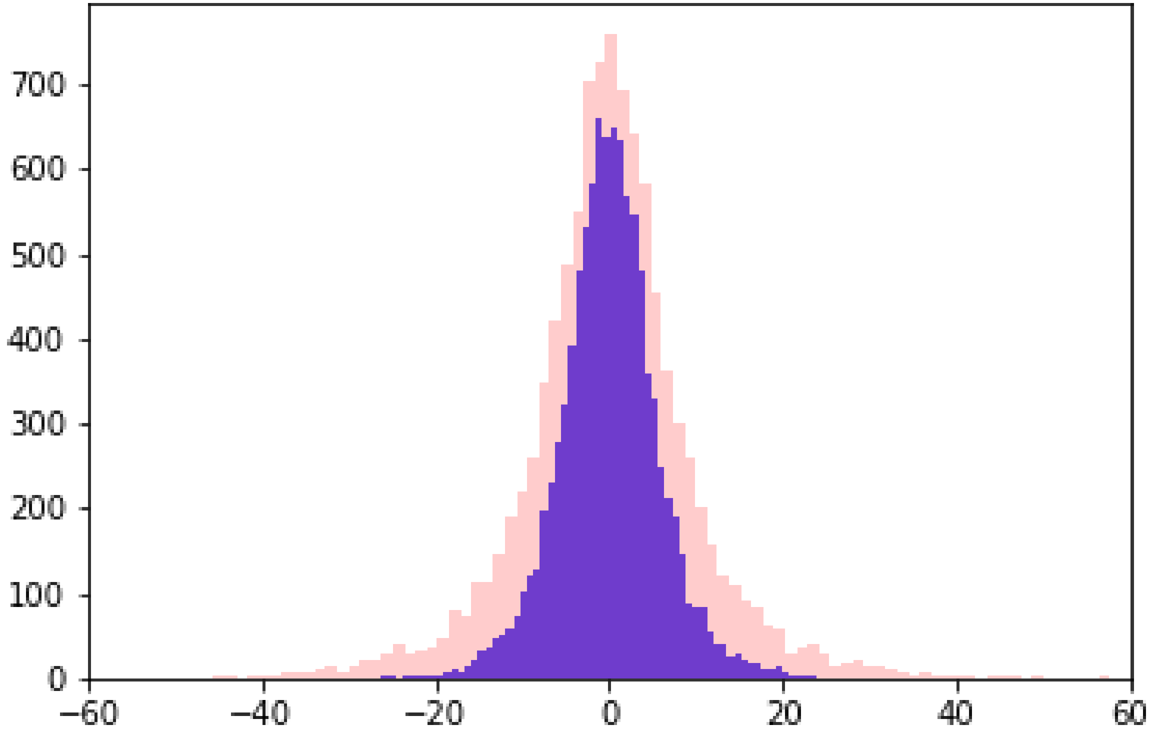
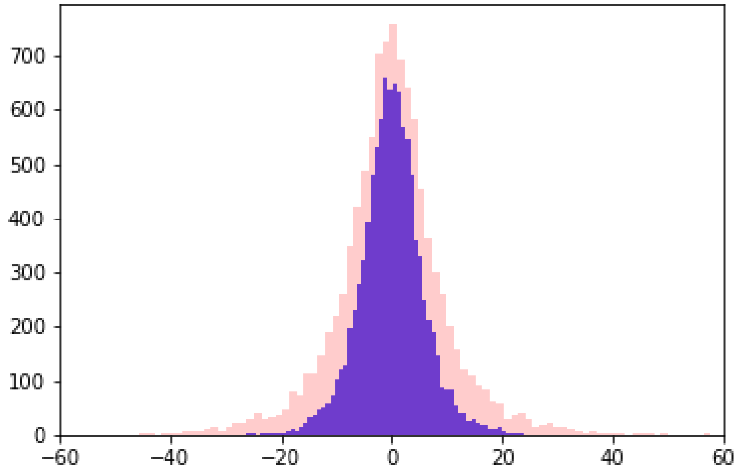
histogram of 10^6 error (units ϵ) simd 1



histogram of 10^6 error (units ϵ) simd 4

Proposition

- Use statistics instead of a single error to judge numerics
- Belief: computational errors has specific distributions, and
- Statistics such as variance has a narrow spread



Validation

1. Use specific input distributions
2. Model distribution of computational errors
3. Match sampled variance to theoretical values for several computational kernels

Validation

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3. Match sampled variance to theoretical values for several computational kernels
    1. SIMD summation
    2. Inner products
    3. Matrix multiplication
    4. Hybrid fixed-float summation

Validation

1. Use specific input distributions
2. Model distribution of computational errors
3. Match sampled variance to theoretical values for several computational kernels
    1. SIMD summation — ***here***
    2. Inner products
    3. Matrix multiplication

    see paper ARITH 2025

    4. Hybrid fixed-float summation — ***here***

Validation

Model Basic Rounding

Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

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If $|x| \in I_k = [2^k, 2^{k+1})$, then $\mathcal{R}_\sigma \sim 2^k\,\mathcal{U}[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, $\epsilon = ulp(1)$.

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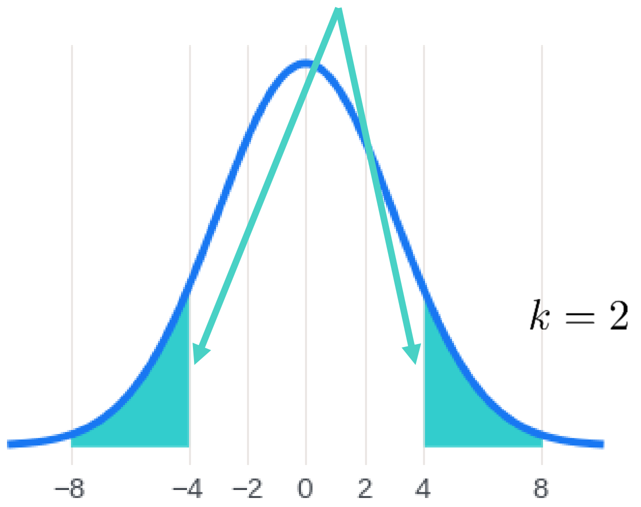
Because $\mathrm{Var}(\mathcal{U}[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]) = \frac{\epsilon^2}{12}$

$$E(\mathcal{R}_\sigma^2) = (\varepsilon^2/12) \sum_k 2^{2k} \boxed{\frac{2}{\sqrt{2\pi}} \int_{2^k}^{2^{k+1}} e^{-t^2/2} dt}$$

and probability of $|x| \in I_k$ is

$$p_k(\sigma) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_{2^k}^{2^{k+1}} e^{-\frac{t^2}{2\sigma^2}} dt$$



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Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

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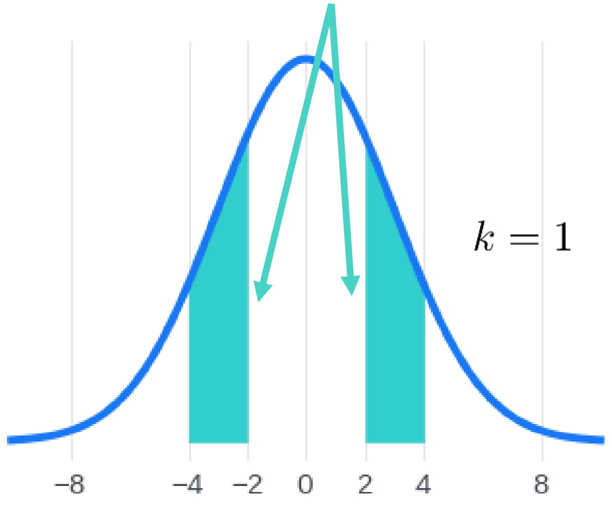

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Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

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and probability of $|x| \in I_k$ is $\quad p_k(\sigma) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_{2^k}^{2^{k+1}} e^{-\frac{t^2}{2\sigma^2}}\, dt$

Thus $E(\mathcal{R}_\sigma^2) = \frac{\epsilon^2}{12} \sum 2^{2k} p_k(\sigma) = \frac{\epsilon^2}{12} F(\sigma) = \frac{\epsilon^2}{12} \sigma^2 F_0(\sigma)$
(easy to see $F_0(2\sigma) = F_0(\sigma)$)

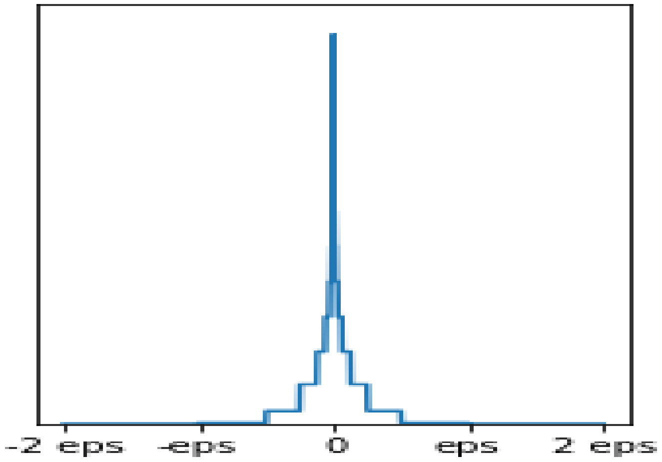
Validation

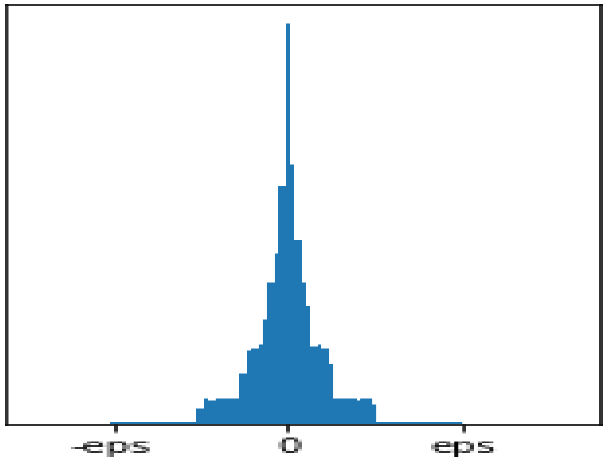
Model Basic Rounding

Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

If $|x| \in I_k = [2^k, 2^{k+1})$, then $\mathcal{R}_\sigma \sim 2^k\,\mathcal{U}[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, $\epsilon = ulp(1)$.

Probability density function and histogram of
$\mathcal{R}_\sigma$, $x \sim \mathcal{N}(0, 1)$, $\epsilon = 2^{-23}$





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Thus $E(\mathcal{R}_\sigma^2) = \frac{\epsilon^2}{12} \sum 2^{2k} p_k(\sigma) = \frac{\epsilon^2}{12} F(\sigma) = \frac{\epsilon^2}{12} \sigma^2 F_0(\sigma)$
(easy to see $F_0(2\sigma) = F_0(\sigma)$)

Similary $E(\mathcal{R}_\sigma^4) = \frac{\epsilon^4}{80} \sum 2^{4k} p_k(\sigma) = \frac{\epsilon^4}{80} G(\sigma) = \frac{\epsilon^4}{80} \sigma^4 G_0(\sigma)$
(easy to see $G_0(2\sigma) = G_0(\sigma)$)

Validation

Model Basic Rounding

Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

$$E(\mathcal{R}_\sigma^2) = \tfrac{\epsilon^2}{12} \sum 2^{2k} p_k(\sigma) = \tfrac{\epsilon^2}{12} F(\sigma) = \tfrac{\epsilon^2}{12} \sigma^2 F_0(\sigma), \qquad F_0(2\sigma) = F_0(\sigma)$$

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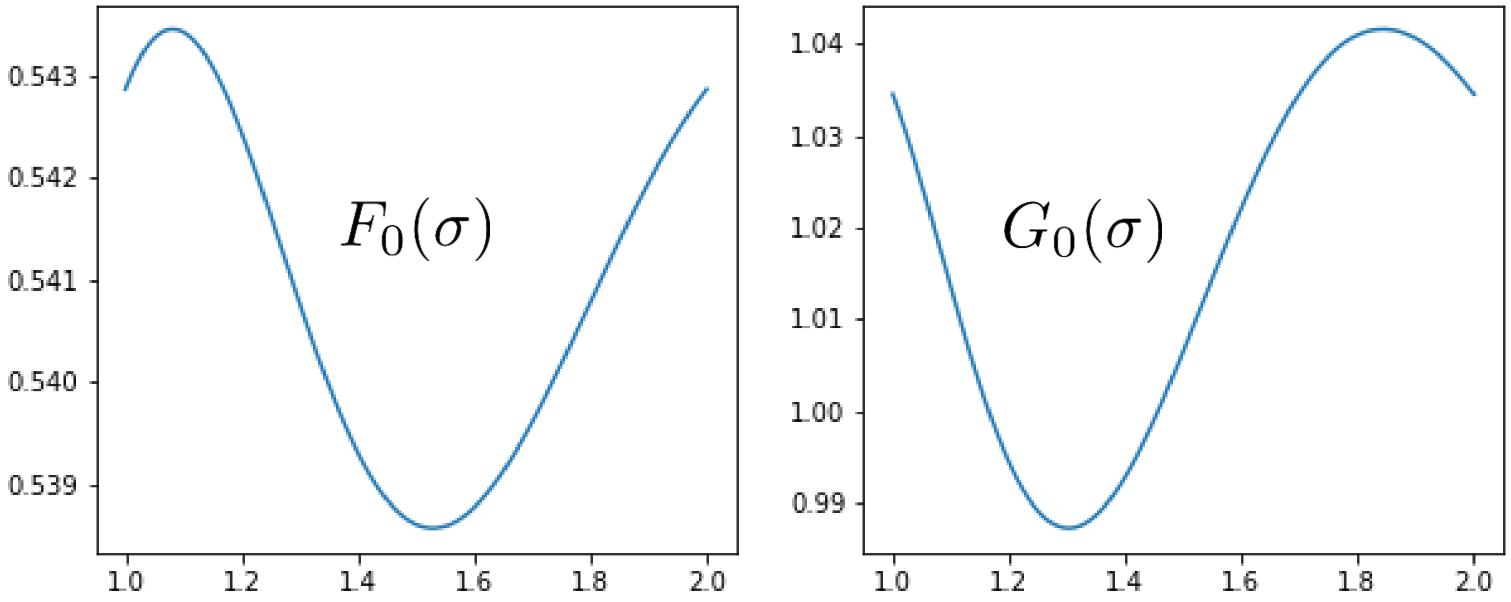
Validation

Model Basic Rounding

Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

$$E(\mathcal{R}_\sigma^2) = \frac{\epsilon^2}{12}\sigma^2 F_0(\sigma) \qquad E(\mathcal{R}_\sigma^4) = \frac{\epsilon^4}{80}\sigma^4 G_0(\sigma)$$



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Model Basic Rounding

Take $x \in \mathbb{R}$, $x \sim \mathcal{N}(0, \sigma^2)$. Define $\mathcal{R}_\sigma := fl(x) - x$.

$E(\mathcal{R}_\sigma^2) = \frac{\epsilon^2}{12}\sigma^2 F_0(\sigma)$ $\qquad$ $E(\mathcal{R}_\sigma^4) = \frac{\epsilon^4}{80}\sigma^4 G_0(\sigma)$

	$F(\sigma)/\sigma^2$	$G(\sigma)/\sigma^4$
c_0	0.54279	1.03445
c_1	0.01827	-0.16159
c_2	-0.13577	-0.85648
c_3	0.16304	4.03253
c_4	0.10800	-2.64662
c_5	-0.26253	-7.00256
c_6	0.10911	13.25400
c_7	0.00000	-8.72614
c_8	0.00000	2.10696

$F_0(\sigma), G_0(\sigma)$ are of the form $\sum_{j=0}^{d} c_j(\sigma - 1)^j$, $1 \leq \sigma \leq 2$

Validation

Model Addition

$$x \sim \mathcal{N}(0, \sigma_x^2) \quad , y \sim \mathcal{N}(0, \sigma_y^2) \quad . \quad \sigma^2 = \sigma_x^2 + \sigma_y^2.$$

$$S_\sigma = fl(x+y) - (x+y)$$

$$E(S_\sigma^2 \; ||x| \in I_k) = \frac{\epsilon^2}{12} 2^{2k}$$

Validation

Model Addition

$$x \sim fl(\,\mathcal{N}(0,\sigma_x^2)\,)\;, y \sim fl(\,\mathcal{N}(0,\sigma_y^2)\,)\;. \quad \sigma^2 = \sigma_x^2 + \sigma_y^2.$$

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Validation

Model Addition

$$x \sim fl(\,\mathcal{N}(0,\sigma_x^2)\,)\,,\, y \sim fl(\,\mathcal{N}(0,\sigma_y^2)\,)\,. \quad \sigma^2 = \sigma_x^2 + \sigma_y^2. \qquad r = \sigma_x^2/\sigma_y^2.$$

$$S_{\sigma\,,r} = fl(x+y) - (x+y)$$

$$E(S^2_{\sigma\,,r} | |x| \in I_k) = \tfrac{\epsilon^2}{12} 2^{2k} \;\; \left(\sum \alpha_{\pm,\ell} P((x/y) \in \pm I_\ell)\right)$$

x/y is a Cauchy distribution, only dependent on r, not σ

Validation

Model Addition

$x \sim fl(\,\mathcal{N}(0,\sigma_x^2)\,)\;, y \sim fl(\,\mathcal{N}(0,\sigma_y^2)\,)\;.\quad \sigma^2 = \sigma_x^2 + \sigma_y^2. \qquad r = \sigma_x^2/\sigma_y^2.$

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x/y is a Cauchy distribution, only dependent on r, not σ

$E(\mathcal{S}^2_{\sigma,r}) = \frac{\epsilon^2}{12}\sigma^2 F_0(\sigma)\phi(r) \qquad E(\mathcal{S}^4_{\sigma,r}) = \frac{\epsilon^4}{80}\sigma^4 G_0(\sigma)\psi(r)$

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$x \sim fl(\mathcal{N}(0,\sigma_x^2))\ ,y \sim fl(\mathcal{N}(0,\sigma_y^2))\ .\quad \sigma^2 = \sigma_x^2 + \sigma_y^2. \qquad r = \sigma_x^2/\sigma_y^2.$

$S_{\sigma,r} = fl(x+y) - (x+y) \qquad E(\mathcal{S}_{\sigma,r}^2) = \frac{\epsilon^2}{12}\sigma^2 F_0(\sigma)\phi(r) \qquad E(\mathcal{S}_{\sigma,r}^4) = \frac{\epsilon^4}{80}\sigma^4 G_0(\sigma)\psi(r)$

	$F(\sigma)/\sigma^2$	$G(\sigma)/\sigma^4$	$\phi(r)$	$\psi(r)$
c_0	0.54279	1.03445	1.00684	0.99967
c_1	0.01827	-0.16159	0.88421	1.94703
c_2	-0.13577	-0.85648	-1.91853	-2.43573
c_3	0.16304	4.03253	-1.93557	5.04192
c_4	0.10800	-2.64662	-0.72735	-16.59014
c_5	-0.26253	-7.00256	0.00000	29.25313
c_6	0.10911	13.25400	0.00000	-24.83812
c_7	0.00000	-8.72614	0.00000	9.10214
c_8	0.00000	2.10696	0.00000	-0.84897

$F_0(\sigma), G_0(\sigma)$ are of the form $\sum_{j=0}^{d} c_j(\sigma-1)^j,\ 1 \le \sigma \le 2$

$\phi(r), \psi(r)$ are of the form $\sum_{j=0}^{d} c_j r^j,\ 0 \le r \le 1.$

Validation

Model SIMD Sum

$x_i \sim fl(\mathcal{N}(0,1))$, $\texttt{simd\_len} = \ell$, $m = L/\ell$

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def sgl_simd_sum(x_data, simd_len):
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$x_i \sim fl(\mathcal{N}(0,1))$, $\texttt{simd\_len} = \ell$, $m = L/\ell$

m inner-loop sum of ℓ values, of the form

$a_0 = y_0$; $a_i = fl(a_{i-1} + y_i)$, $i = 1, 2, \ldots, \ell - 1$

Each error is $\mathcal{S}_{\sigma_i, r_i}$, $\sigma_i = \sqrt{1+i}$, $r_i = 1/i$

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1 outer-loop sum of m values, of the form

$a_0 = y_0;\ a_i = fl(a_{i-1} + y_i),\ i = 1, 2, \ldots, m - 1$

Each error is $\mathcal{S}_{\sigma_i, r_i}$, $\sigma_i = \sqrt{(1+i)\ell}$, $r_i = 1/i$

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Model SIMD Sum

$x_i \sim fl(\mathcal{N}(0,1))$, $\texttt{simd\_len} = \ell$, $m = L/\ell$

Inner-loop error $\mathcal{S}_{\sigma_i, r_i}$, $\sigma_i = \sqrt{1+i}$, $r_i = 1/i$

$i = 1, 2, \ldots, \ell - 1$, m sets

Outer-loop error $\mathcal{S}_{\sigma_i, r_i}$, $\sigma_i = \sqrt{(1+i)\ell}$, $r_i = 1/i$

$i = 1, 2, \ldots, \ell - 1$, one set

Total error $\texttt{SIMD} - \ell$ sum: Δ_ℓ

$\Delta_{(\ell)} = \tau_1 + \tau_2 + \cdots + \tau_{L-1}$

τ_i, τ_j independent and $E(\tau_i) = 0$

$$E(\Delta_\ell^2) = \frac{\epsilon^2}{12}\left(m \sum_{i=1}^{\ell-1} F(\sqrt{1+i})\phi(1/i) + \sum_{i=1}^{m-1} F(\sqrt{1+i})\phi(1/i)\right)$$

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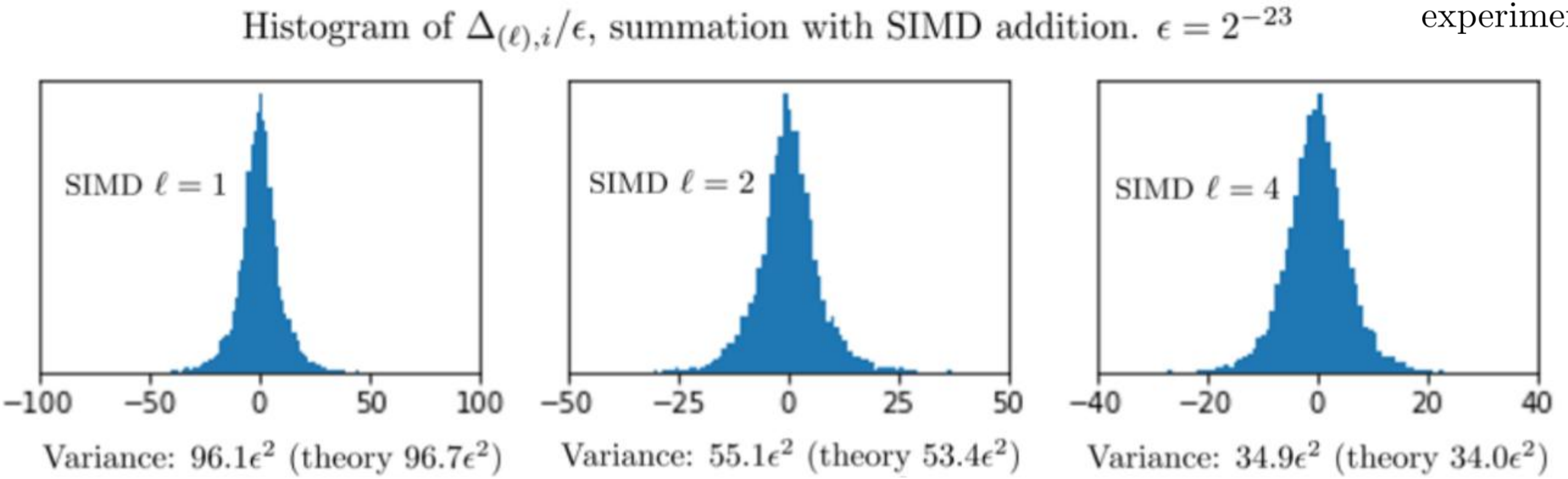
Validation

Model SIMD Sum

$N = 10^4$
summation
experiments

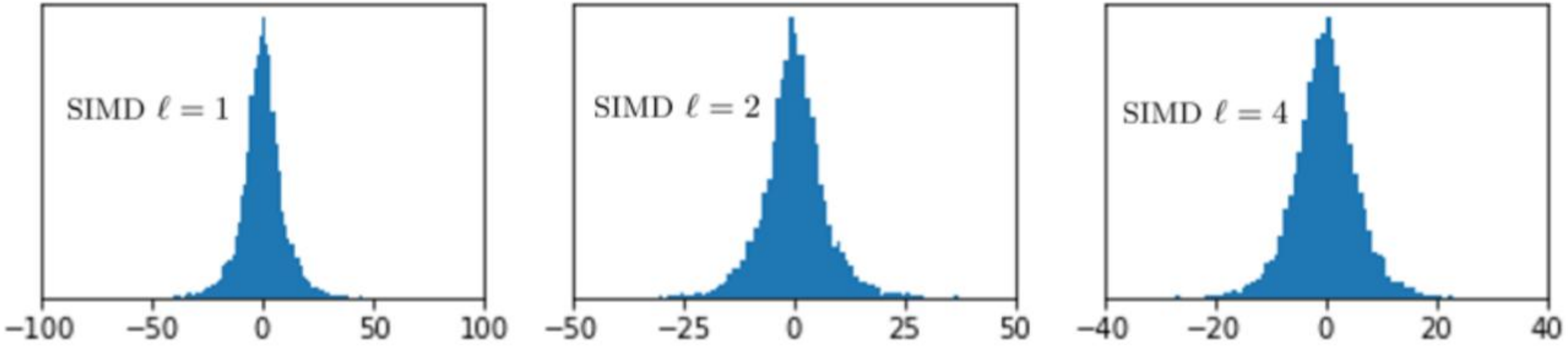


Validation

Model SIMD Sum

$N = 10^4$ summation experiments

Histogram of $\Delta_{(\ell),i}/\epsilon$, summation with SIMD addition. $\epsilon = 2^{-23}$



Variance: $96.1\epsilon^2$ (theory $96.7\epsilon^2$) Variance: $55.1\epsilon^2$ (theory $53.4\epsilon^2$) Variance: $34.9\epsilon^2$ (theory $34.0\epsilon^2$)

is the difference reasonable?

Validation

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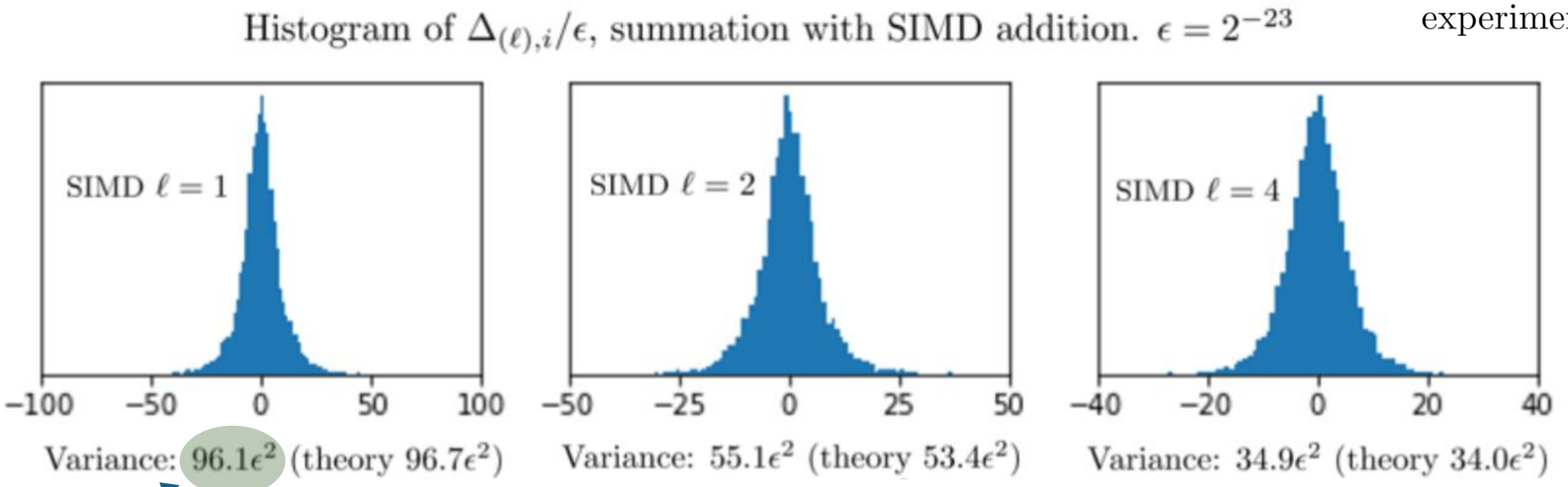

One sample of $Z = \sum_{i=1}^{N} \Delta^2_{(\ell),i}/N$

$\Delta_{(\ell)} = \tau_1 + \tau_2 + \cdots + \tau_{L-1}$

Validation

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$N = 10^4$
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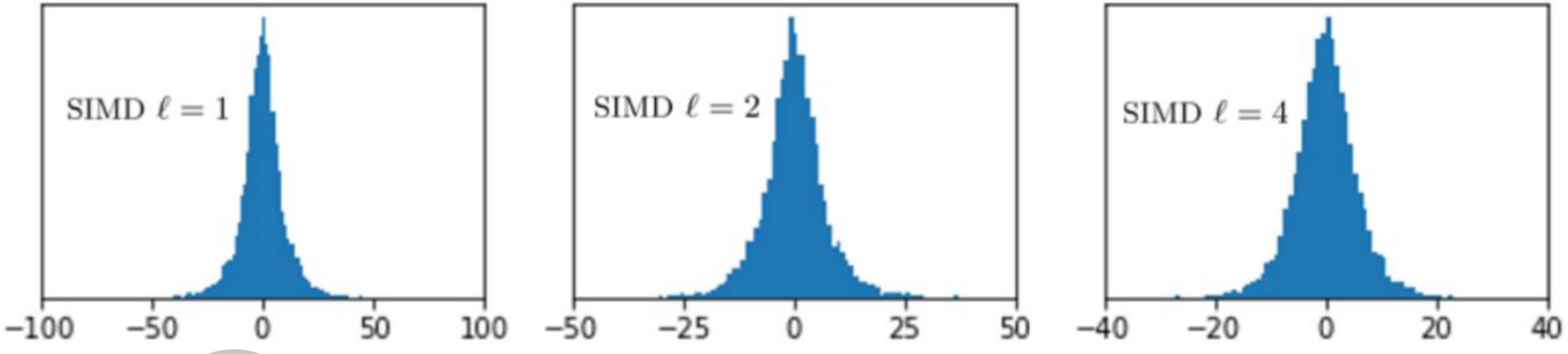
Histogram of $\Delta_{(\ell),i}/\epsilon$, summation with SIMD addition. $\epsilon = 2^{-23}$



Variance: $96.1\epsilon^2$ (theory $96.7\epsilon^2$) Variance: $55.1\epsilon^2$ (theory $53.4\epsilon^2$) Variance: $34.9\epsilon^2$ (theory $34.0\epsilon^2$)

One sample of $Z = \sum_{i=1}^{N} \Delta^2_{(\ell),i}/N \sim \mathrm{Var}(\Delta_{(\ell)}) + \frac{\sigma}{\sqrt{N}}\mathcal{N}(0,1)$ $\quad \sigma^2 = \mathrm{Var}(\Delta^2_{(\ell)})$

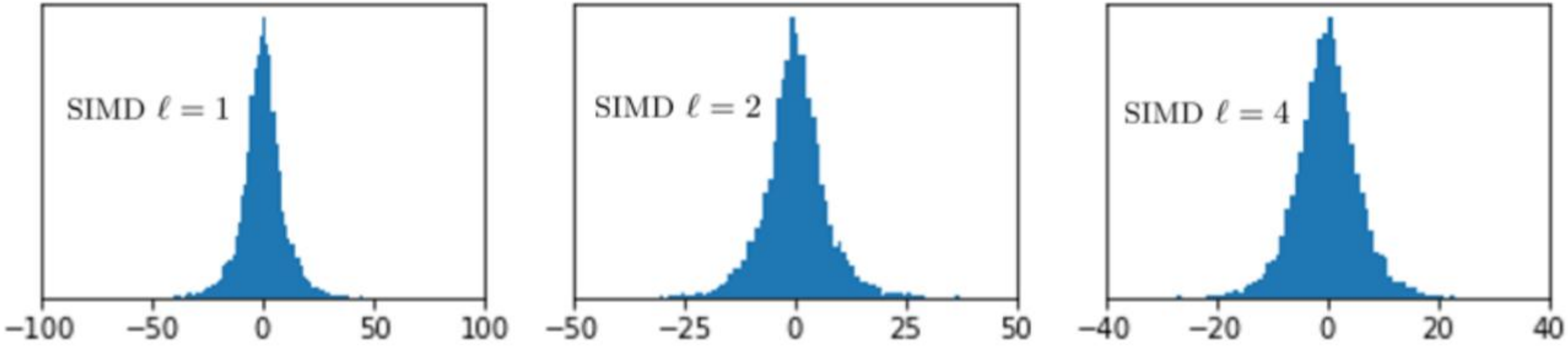
$\Delta_{(\ell)} = \tau_1 + \tau_2 + \cdots + \tau_{L-1}$

Validation

Model SIMD Sum

$N = 10^4$
summation
experiments

Histogram of $\Delta_{(\ell),i}/\epsilon$, summation with SIMD addition. $\epsilon = 2^{-23}$



Variance: $96.1\epsilon^2$ (theory $96.7\epsilon^2$) Variance: $55.1\epsilon^2$ (theory $53.4\epsilon^2$) Variance: $34.9\epsilon^2$ (theory $34.0\epsilon^2$)

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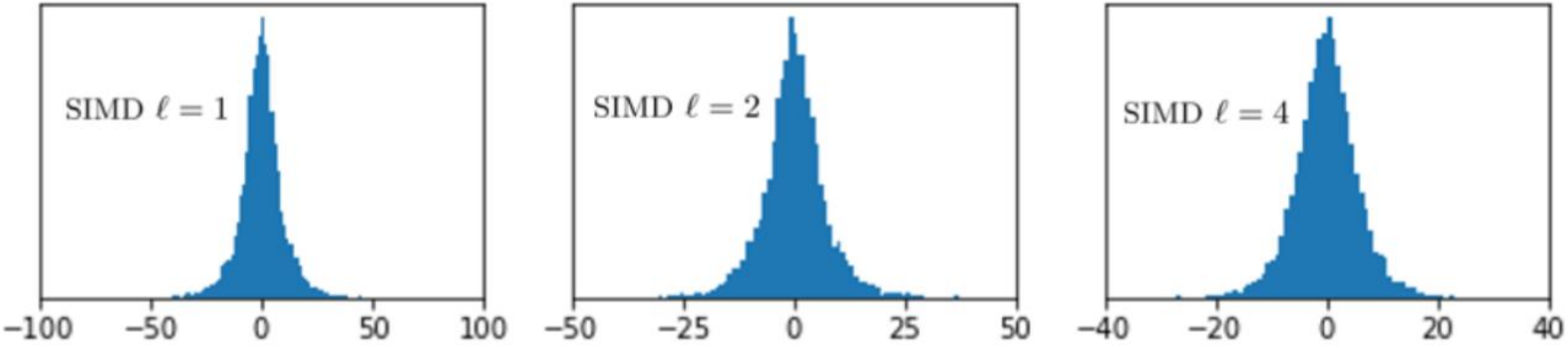
$\Delta_{(\ell)} = \tau_1 + \tau_2 + \cdots + \tau_{L-1}$

$$\mathrm{Var}(\Delta^2_{(\ell)}) = E(\Delta^4_{(\ell)}) - E^2(\Delta^2_{(\ell)}) = \sum_i ((E(\tau_i^4) - E^2(\tau_i^2)) + 10\sum_{i<j} E(\tau_i^2)E(\tau_j^2)$$

Validation

Model SIMD Sum

Histogram of $\Delta_{(\ell),i}/\epsilon$, summation with SIMD addition. $\epsilon = 2^{-23}$



Variance: $96.1\epsilon^2$ (theory $96.7\epsilon^2$)
standard deviation: $2.2\epsilon^2$

Variance: $55.1\epsilon^2$ (theory $53.4\epsilon^2$)
standard deviation $1.2\epsilon^2$

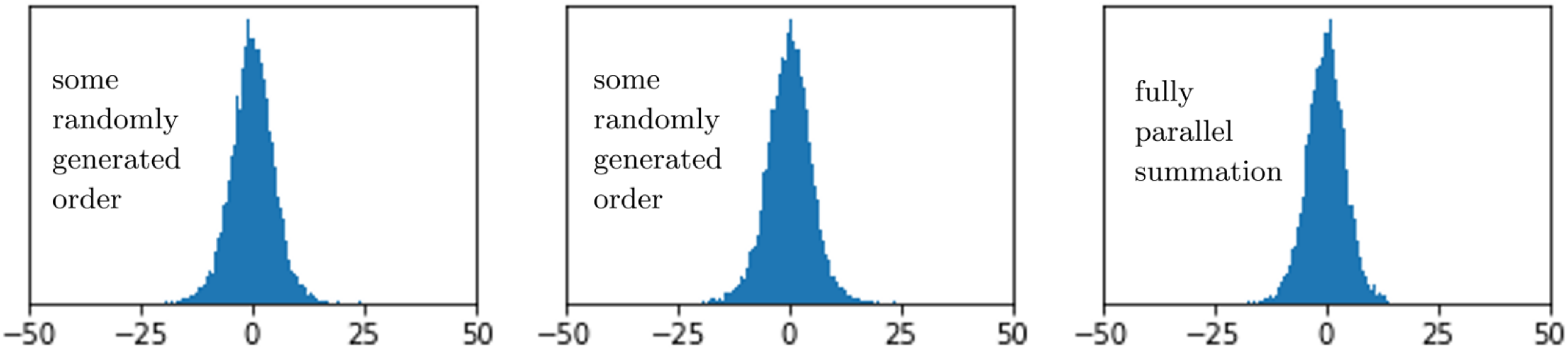
Variance: $34.9\epsilon^2$ (theory $34.0\epsilon^2$)
standard deviation $0.8\epsilon^2$

One sample of $Z = \sum_{i=1}^{N} \Delta^2_{(\ell),i}/N \sim \mathrm{Var}(\Delta_{(\ell)}) + \frac{\sigma}{\sqrt{N}}\mathcal{N}(0,1) \qquad \sigma^2 = \mathrm{Var}(\Delta^2_{(\ell)})$

Validation

General Tree Sum

Histogram of $\Delta_{(\ell),i}/\epsilon$, general summation with different orders. $\epsilon = 2^{-23}$



Variance: $24.5\epsilon^2$ (theory $24.0\epsilon^2$)
standard deviation: $0.54\epsilon^2$

Variance: $25.8\epsilon^2$ (theory $25.5\epsilon^2$)
standard deviation $0.57\epsilon^2$

Variance: $20.2\epsilon^2$ (theory $20.4\epsilon^2$)
standard deviation $0.46\epsilon^2$

Validation

Model Fixed-Float Addition

Validation

Model Fixed-Float Addition

- Matrix multiplication is a common target for acceleration

Validation

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- Summation is often a critical path if pairwise rounding is needed

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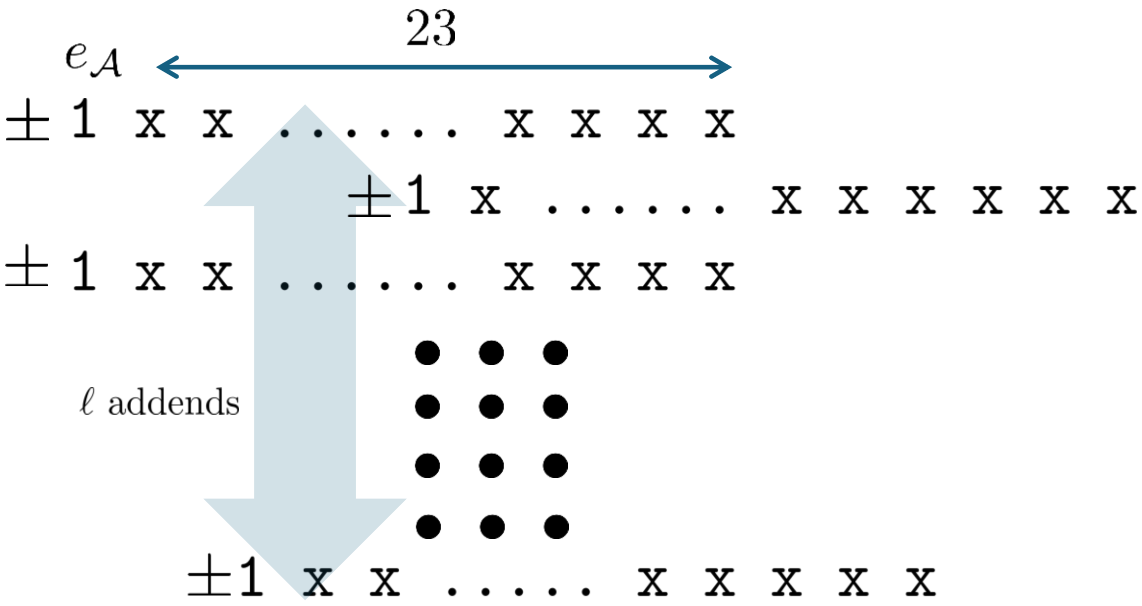
Validation

Model Fixed-Float Addition

- Matrix multiplication is a common target for acceleration
- Summation is often a critical path if pairwise rounding is needed
- Hybrid fixed-float summation is an obvious alternative
- A hypothetical FP32 architecture:
  - For ℓ input fp numbers, align with the largest exponent
  - Smaller exponent values are right shifted
  - Bits beyond $23 + d$ fractional values are rounded off
  - Sum the rounded values exactly; then round to FP32

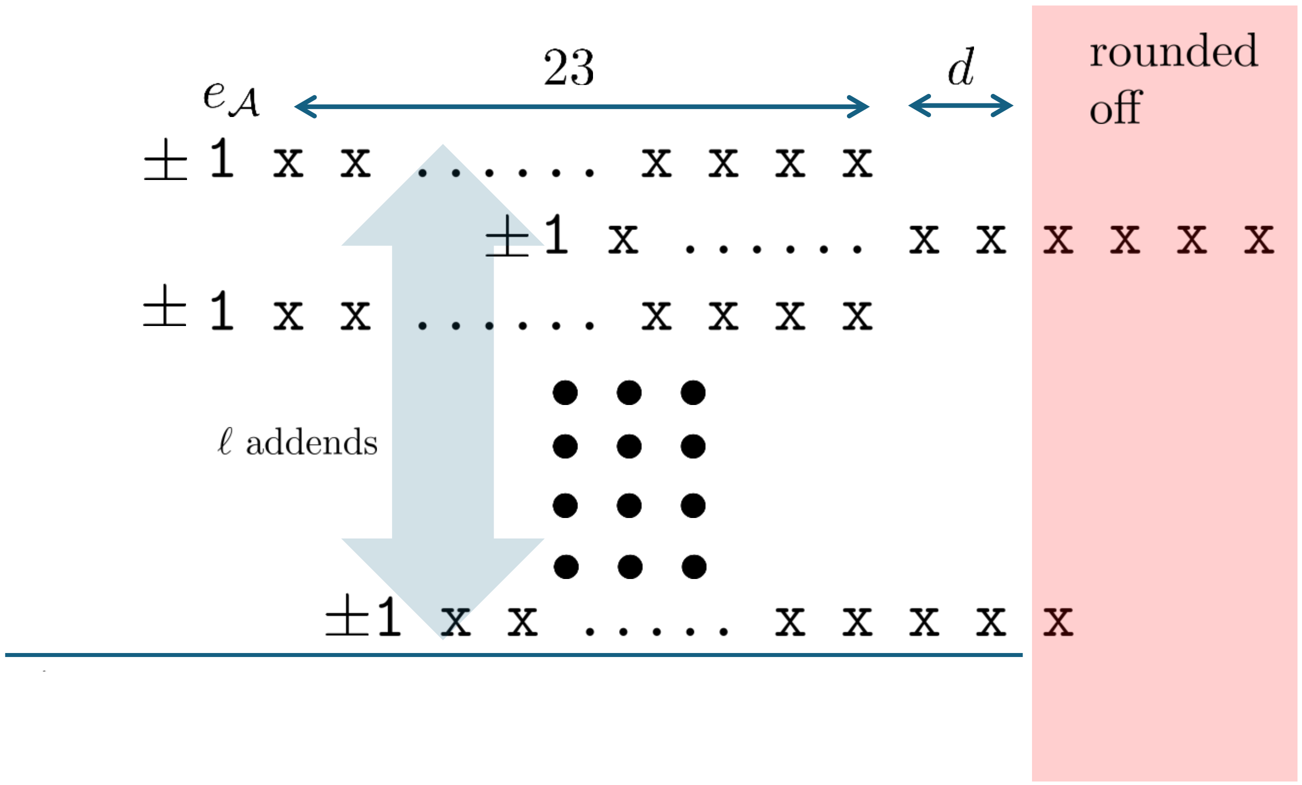
Validation

Model Fixed-Float Addition



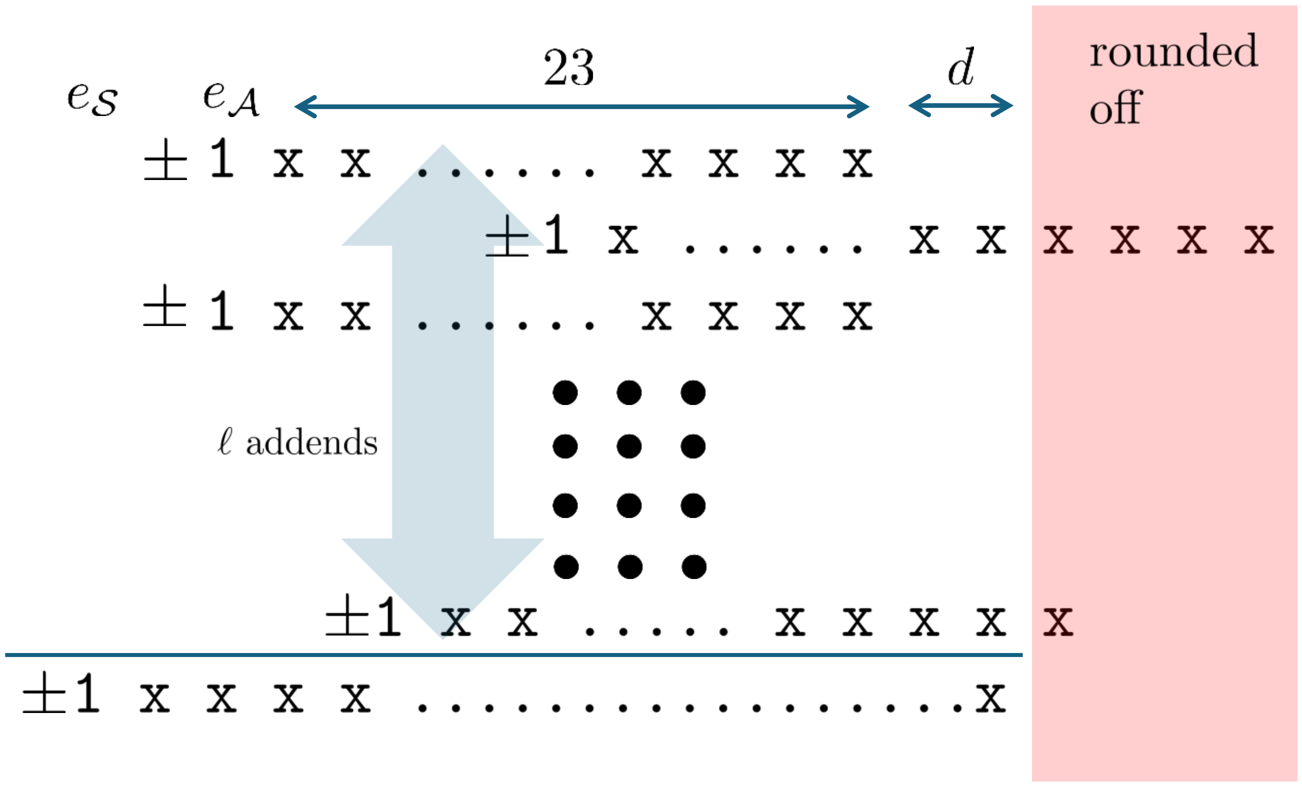
Validation

Model Fixed-Float Addition



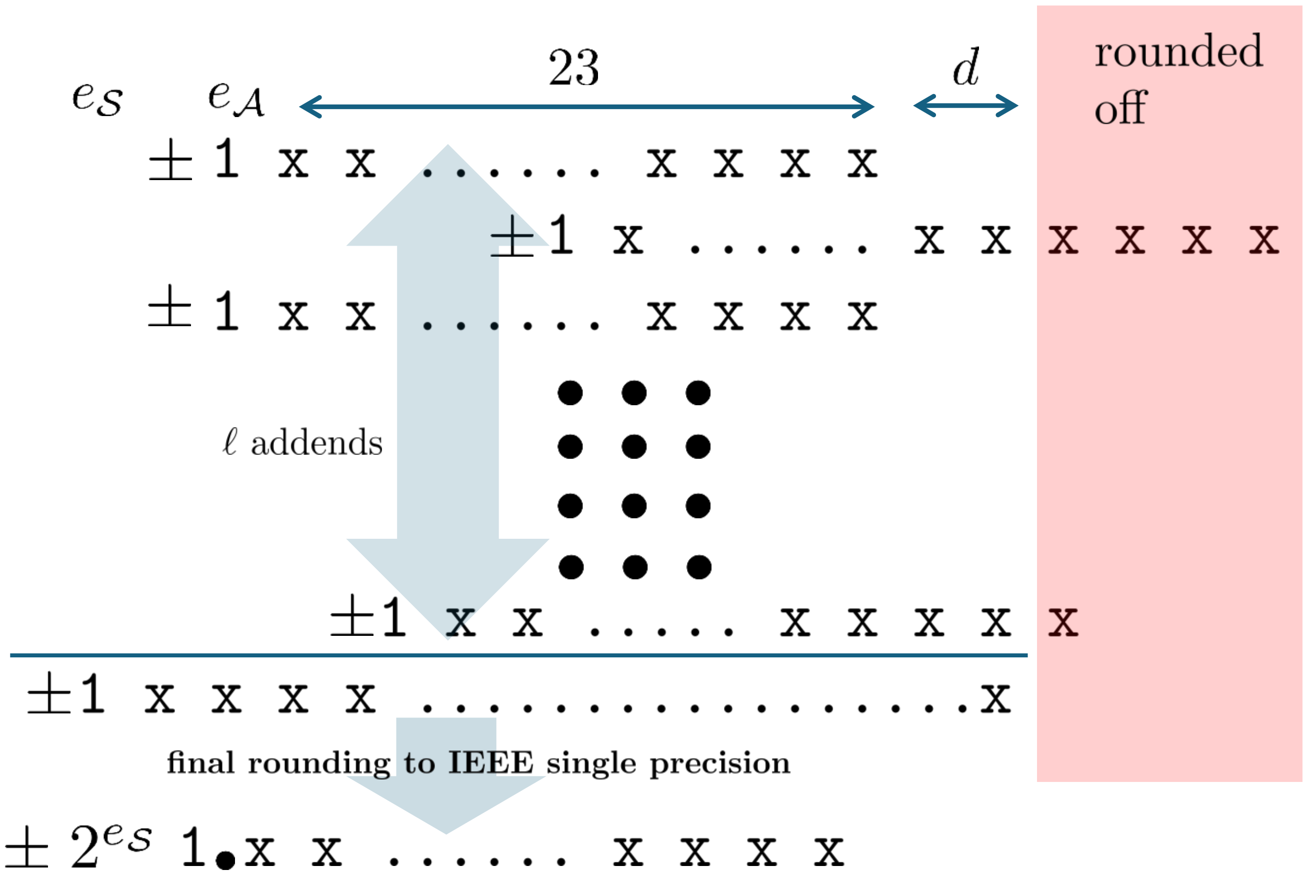
Validation

Model Fixed-Float Addition



Validation

Model Fixed-Float Addition



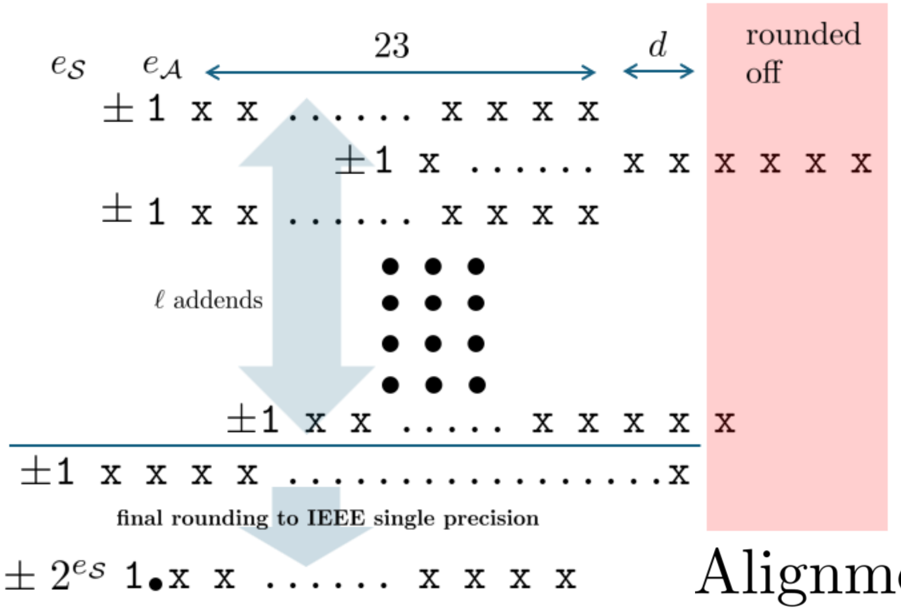
Validation

Model Fixed-Float Addition

$\Delta_{\mathcal{A}}$ denotes alignment error of one summand

v_j is variance of rounding j bits
($v_1 = 1/8$, $v_2 = 3/32$, etc, $\to 1/12$)

$$E(\Delta_{\mathcal{A}}^2 \mid \text{alignment expo } e_a, \text{ round } j \text{ bits})$$
$$= 2^{-2(23+d)} \qquad 2^{2e_a} \qquad v_j$$



Conversion Error

Alignment Error

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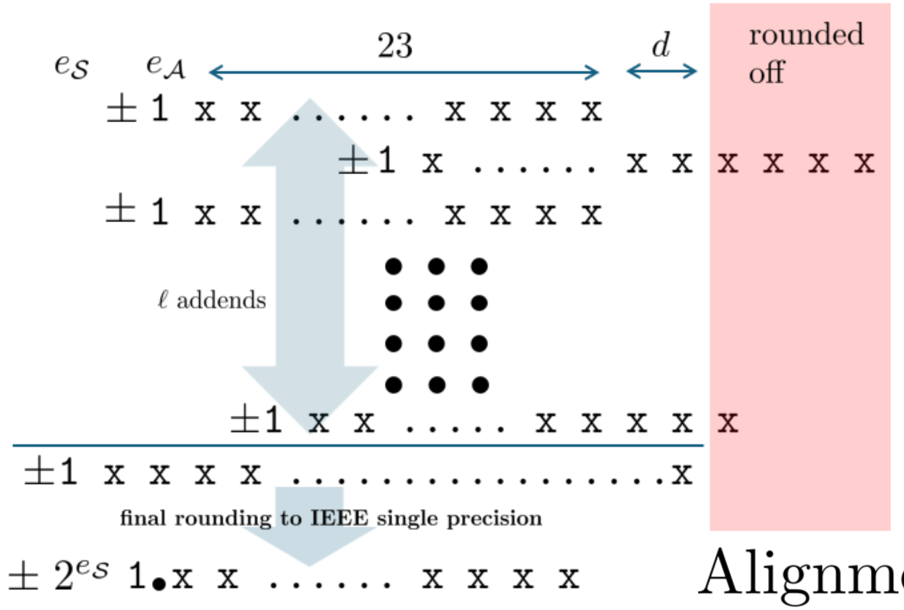
$$E(\Delta_{\mathcal{A}}^2)$$
$$= 2^{-2(23+d)} \sum_{e_a} \sum_{j>1} 2^{2e_a} p_j(e_a) v_j$$



Conversion Error

Alignment Error

$p_j(e_a)$ is probability that
- alignment exponent of $\ell - 1$ values is e_a
- and j bits are to be rounded off

Validation

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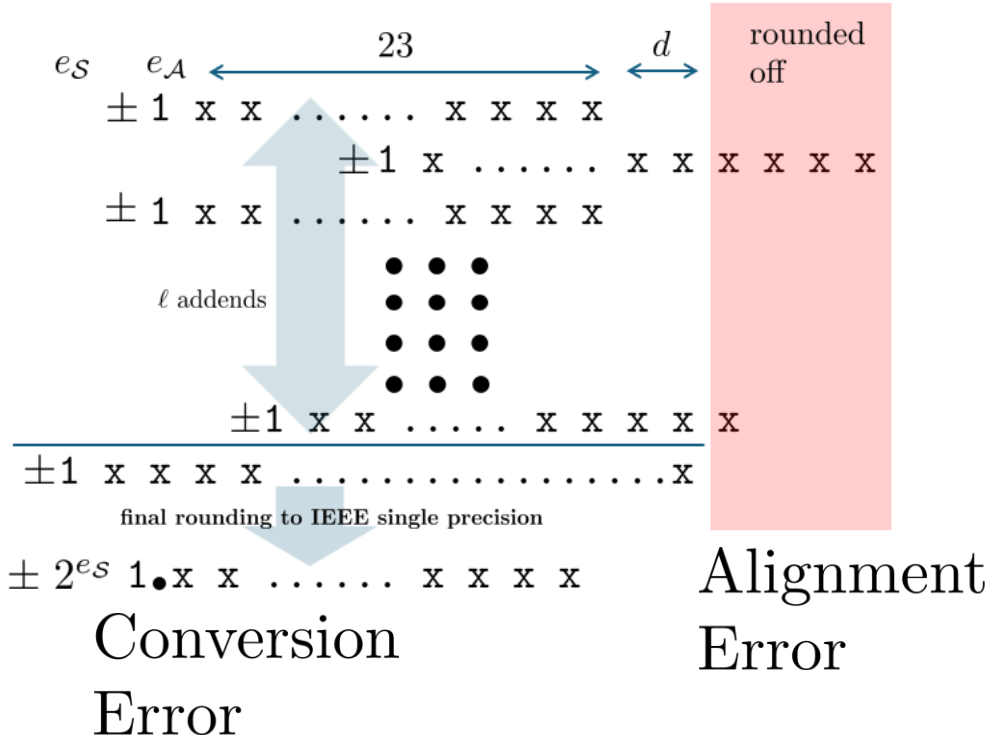

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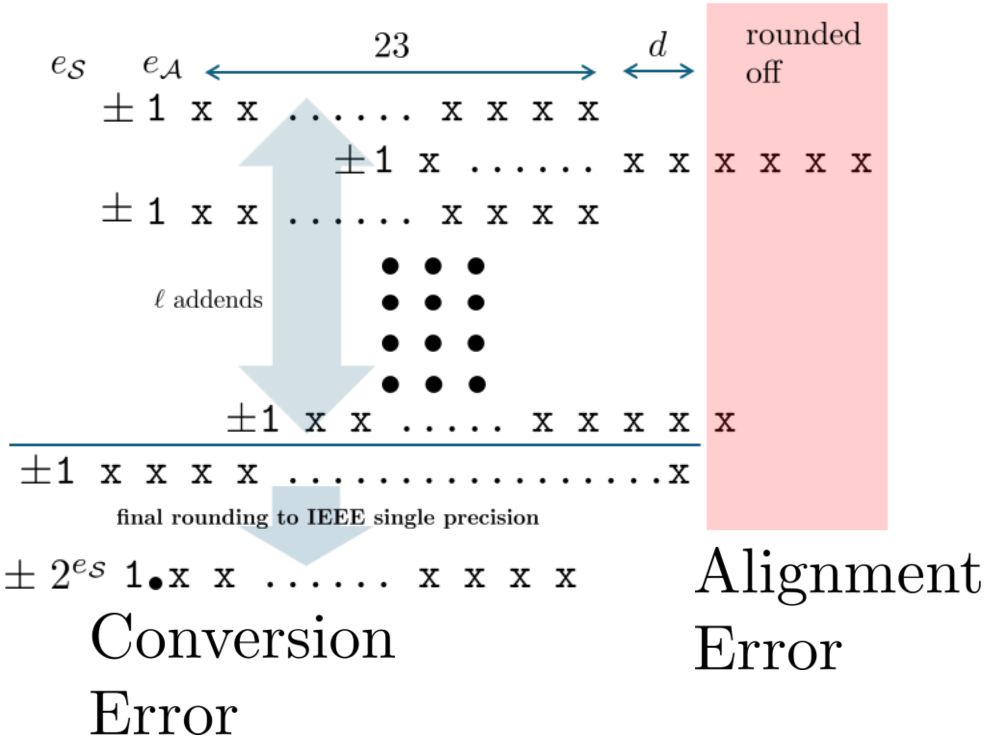
$$p_j(e_a) = \text{Prob(align. expo of } \ell - 1 \text{ elem. is } e_a) \times \pi_{e_a - d - j}$$

Let $\pi_k = \text{Prob}(|x| \in I_k)$ $\quad \hat{\pi}_k = \text{Prob}(|x| \in I_j, j < k)$



Validation

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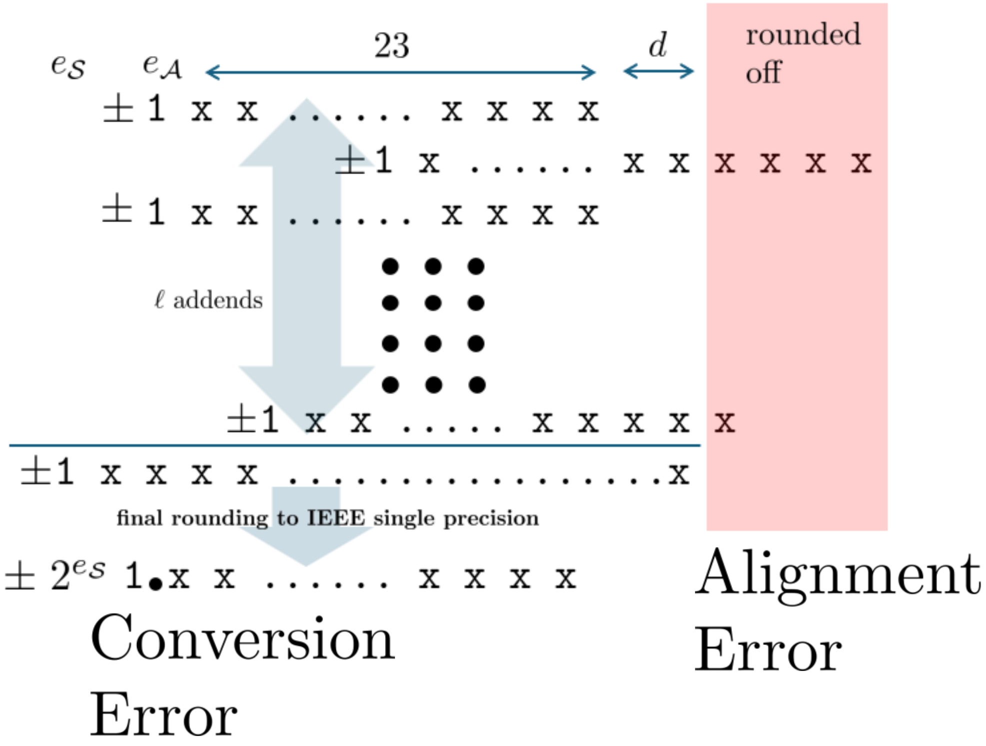
- and j bits are to be rounded off

$$p_j(e_a) = \underbrace{\text{Prob(align. expo of } \ell - 1 \text{ elem. is } e_a)}_{\lambda_{\ell-1,e_a}} \times \pi_{e_a-d-j}$$

Let $\pi_k = \text{Prob}(|x| \in I_k)$ $\quad \hat{\pi}_k = \text{Prob}(|x| \in I_j, j < k)$

$$\lambda_{\ell-1,e_a} = \sum_{i=1}^{\ell-1} \binom{\ell-1}{i} \pi_{e_a}^i \hat{\pi}_{e_a}^{\ell-1-i} = (\pi_{e_a} + \hat{\pi}_{e_a})^{\ell-1} - \hat{\pi}_{e_a}^{\ell-1}$$



Validation

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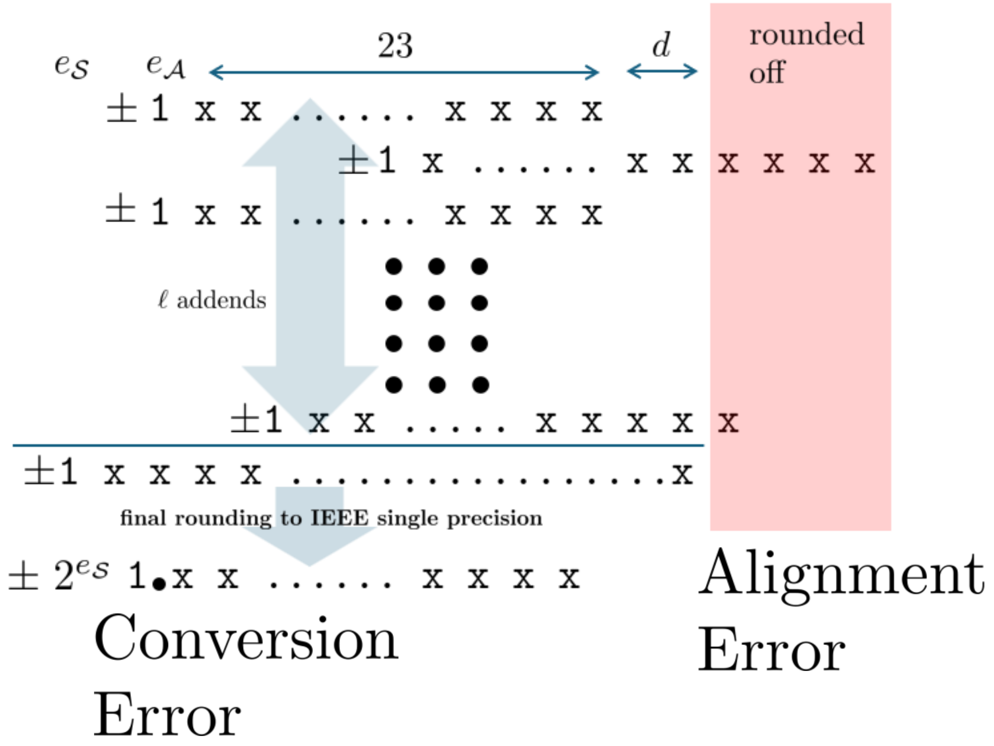
($v_1 = 1/8$, $v_2 = 3/32$, etc, $\to 1/12$)

$E(\Delta_{\mathcal{A}}^2)$
$$= 2^{-2(23+d)} \sum_{e_a} \sum_{j>1} 2^{2e_a} p_j(e_a) v_j$$

$E(\Delta_{\mathcal{A}}^4)$
$$= 2^{-4(23+d)} \sum_{e_a} \sum_{j>1} 2^{4e_a} p_j(e_a) w_j$$



$p_j(e_a)$ is probability that
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- and j bits are to be rounded off

Validation

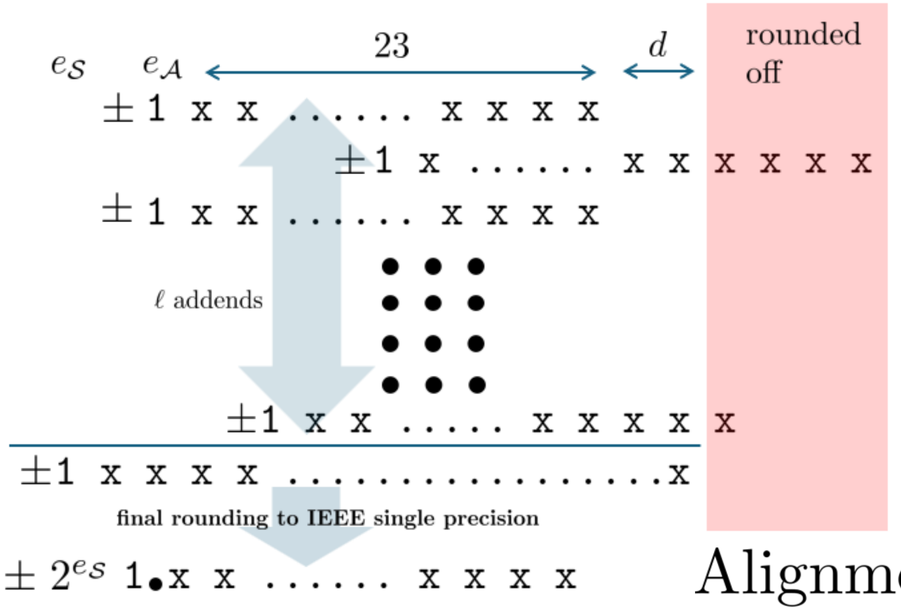
Model Fixed-Float Addition

$\Delta_{\mathcal{C}}$ denotes conversion error of final sum

v_j is variance of rounding j bits
$(v_1 = 1/8,\ v_2 = 3/32,\ \text{etc},\ \to 1/12)$

$E(\Delta_{\mathcal{C}}^2 \mid \text{alignment expo } e_a, \text{ round } j \text{ bits})$

$= 2^{-2(23+d)} \qquad 2^{2(e_a+j)} \qquad v_j$



Conversion Error

Alignment Error

Validation

Model Fixed-Float Addition

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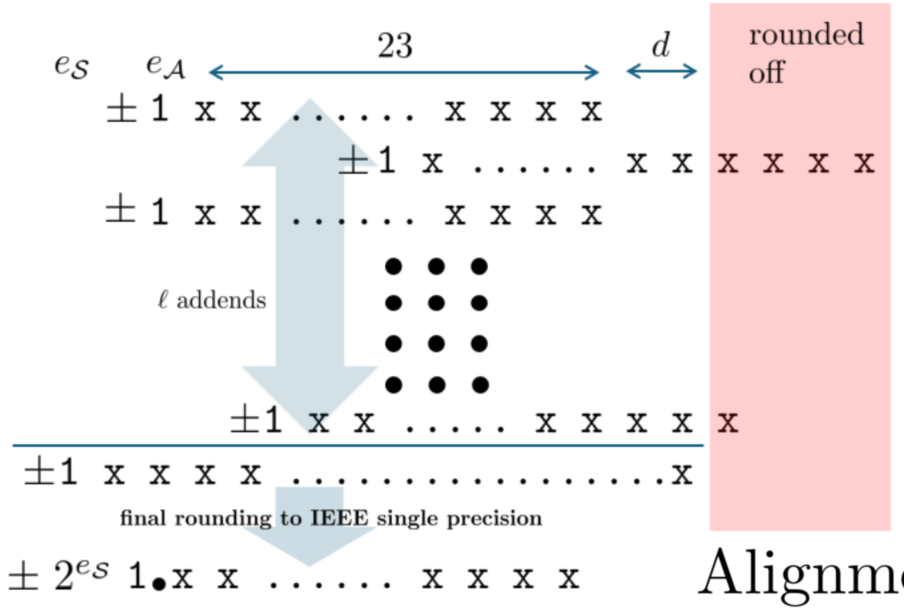
$$E(\Delta_{\mathcal{C}}^2)$$
$$= 2^{-2(23+d)} \sum_{e_a} \sum_{j=1}^{\lceil \log_2(d) \rceil + d} 2^{2(e_a+j)} q_j(e_a) v_j$$



Conversion Error

Alignment Error

$q_j(e_a)$ is probability that
- alignment exponent of $\ell - 1$ values is e_a
- and j bits are to be rounded off

Validation   Model Fixed-Float Addition

$\Delta_{\mathcal{C}}$ denotes conversion error of final sum

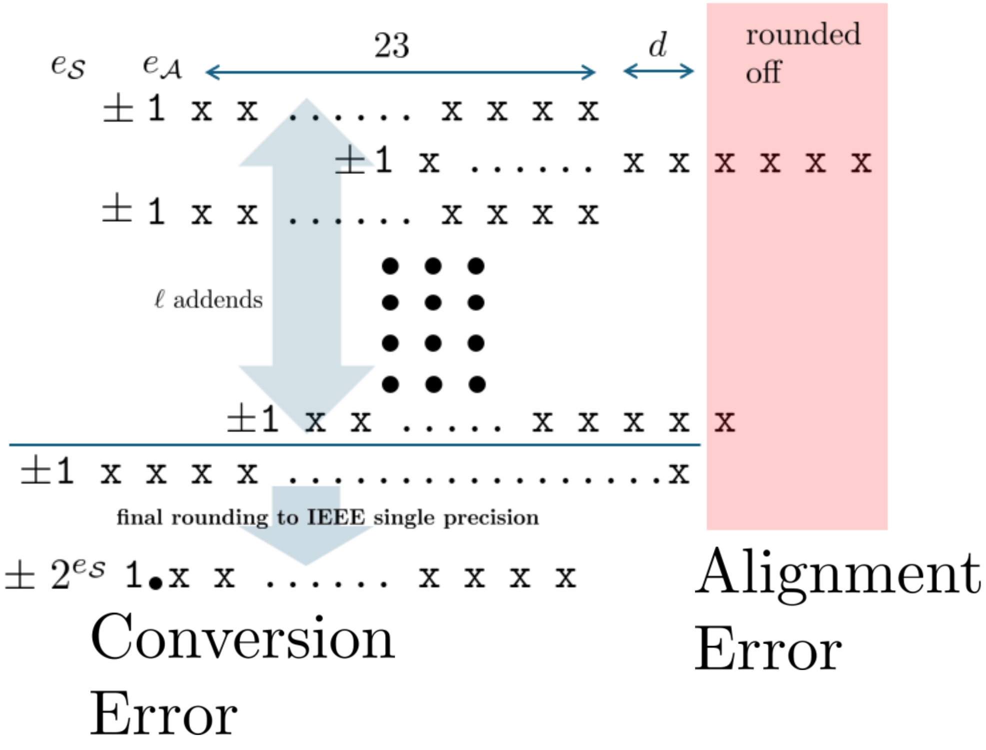
v_j is variance of rounding j bits

$$E(\Delta_{\mathcal{C}}^2) = 2^{-2(23+d)} \sum_{e_a} \sum_{j=1}^{\lceil \log_2(d) \rceil + d} 2^{2(e_a+j)} q_j(e_a) v_j$$

$q_j(e_a)$ is probability that
- alignment exponent of $\ell - 1$ values is e_a
- and j bits are to be rounded off

$$q_j(e_a) = \int_{I_{j+e_a-d}} g_{e_a}(t)\, dt$$

$g_{e_a}(t)$ is the density of the sum of ℓ iids in $\mathcal{N}(0,1)$ where align. expo. is e_a



Validation

Model Fixed-Float Addition

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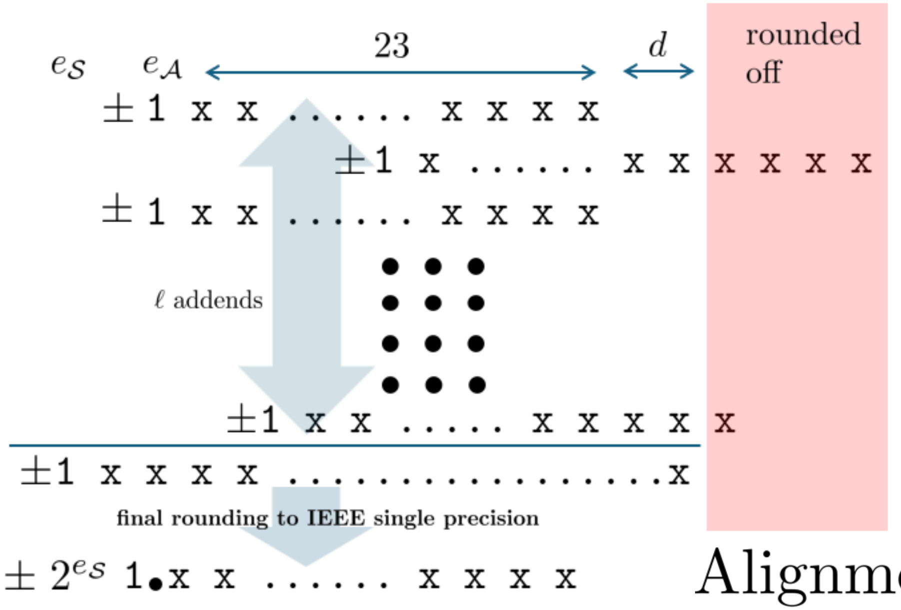

Conversion Error

Alignment Error

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$$g_{e_a}(t) = \sum_{i=1}^{\ell} \binom{\ell}{i} (\overbrace{\rho_{e_a} \star \cdots \star \rho_{e_a}}^{i-\text{times}})(\overbrace{\tilde{\rho}_{e_a} \star \cdots \star \tilde{\rho}_{e_a}}^{(\ell-i)-\text{times}})(t)$$

$\tilde{\rho}_{e_a}(t)$: density of $x \in I_j$, $j < e_a$

Validation

Model Fixed-Float Addition

$\Delta_{\mathcal{C}}$ denotes conversion error of final sum

v_j is variance of rounding j bits

$$E(\Delta_{\mathcal{C}}^2) = 2^{-2(23+d)} \sum_{e_a} \sum_{j=1}^{\lceil \log_2(d) \rceil + d} 2^{2(e_a+j)} q_j(e_a) v_j$$

$q_j(e_a)$ is probability that

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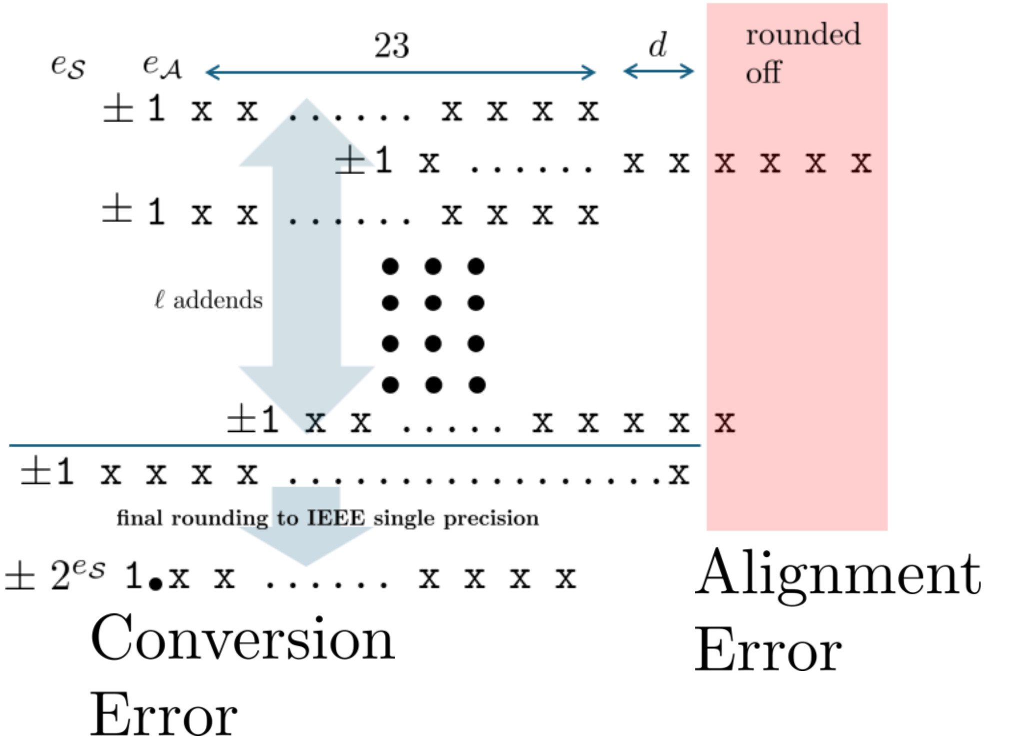
$$q_j(e_a) = \int_{I_{j+e_a-d}} g_{e_a}(t)\, dt$$



$g_{e_a}(t)$ is the density of the sum of ℓ iids in $\mathcal{N}(0,1)$ where align. expo. is e_a

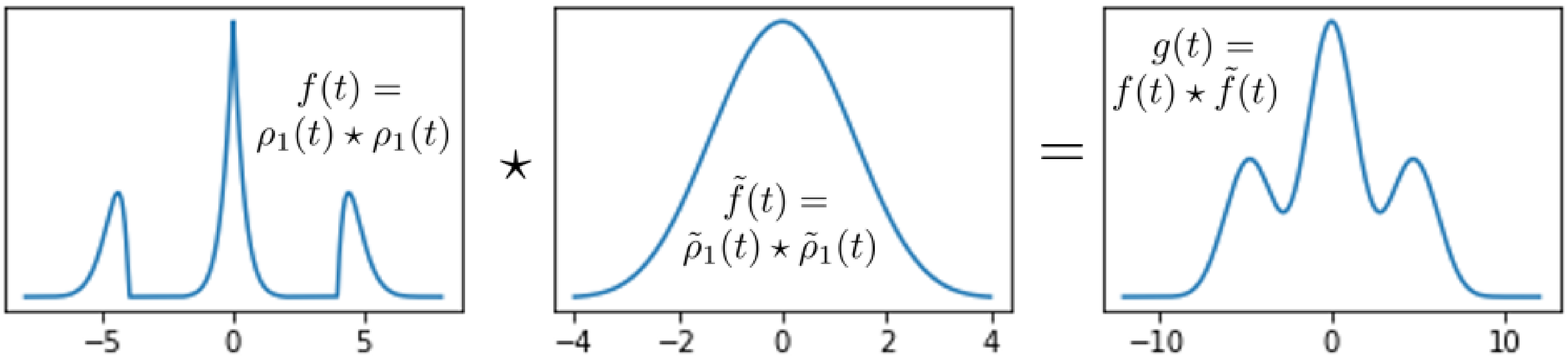
$$g_{e_a}(t) = \sum_{i=1}^{\ell} \binom{\ell}{i} (\overbrace{\rho_{e_a} \star \cdots \star \rho_{e_a}}^{i-\text{times}})(\overbrace{\tilde{\rho}_{e_a} \star \cdots \star \tilde{\rho}_{e_a}}^{(\ell-i)-\text{times}})(t)$$

$\tilde{\rho}_{e_a}(t)$: density of $x \in I_j$, $j < e_a$

Example of $\ell = 4$, $e_a = 1$

$$\rho_1(t) \star \rho_1(t) \star \tilde{\rho}_1(t) \star \tilde{\rho}_1(t)$$

Density of $x_1 + x_2 + x_3 + x_4$; $x_j \sim \mathcal{N}(0,1)$, and $|x_{1,2}| \in [2,4)$, $|x_{3,4}| < 2$



Validation

Model Fixed-Float Addition

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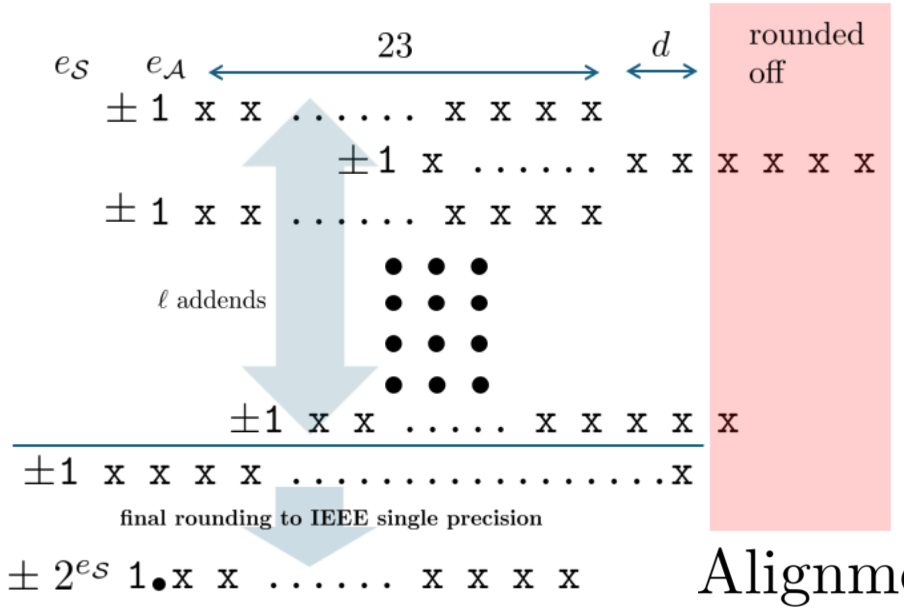

Conversion Error

Alignment Error

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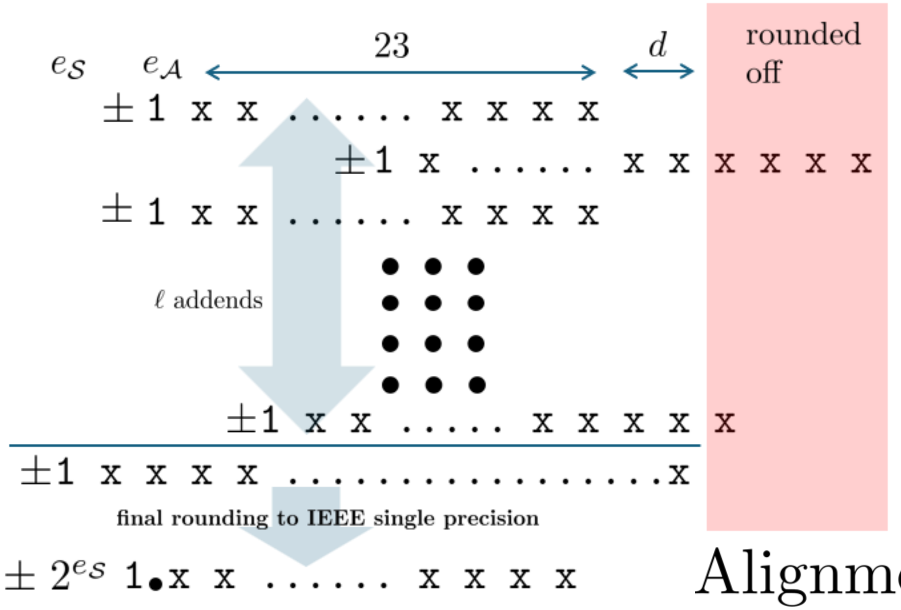
v_j is variance of rounding j bits
($v_1 = 1/8$, $v_2 = 3/32$, etc, $\to 1/12$)

$$E(\Delta_{\mathcal{C}}^2) = 2^{-2(23+d)} \sum_{e_a} \sum_{j=1}^{\lceil \log_2(d) \rceil + d} 2^{2(e_a+j)} q_j(e_a) v_j$$

$$E(\Delta_{\mathcal{C}}^4) = 2^{-4(23+d)} \sum_{e_a} \sum_{j=1}^{\lceil \log_2(d) \rceil + d} 2^{4(e_a+j)} q_j(e_a) w_j$$



Conversion Error

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Validation

Model Fixed-Float Addition

	Mean Square Error (Variance): Sampled, Model, Deviation								
	$\ell = 4$			$\ell = 8$			$\ell = 16$		
Alignment	3.18e-17	3.01e-17	1.4	5.24e-17	5.31e-17	0.4	8.16e-17	8.30e-17	0.6
Conversion	2.68e-15	2.73e-15	0.7	5.27e-15	5.32e-15	0.4	1.04e-14	1.06e-14	0.5
Fixed-Float Accumulator	2.82e-15	2.85e-15	0.4	5.81e-15	5.75e-15	0.5	1.20e-14	1.19e-14	0.4

Validation

- Statistics of rounding errors are reliable metrics
- The main point of the modeling here is that we **do not** need to do it
- Statistics collected by sampling on reference numerical kernels suffices

Applications

- Set error thresholds that are more theoretically supported
- Numerical diagnostics
- Design explorations