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An arithmetical viewpoint on conversion theorems

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From decimals to continued fractions

- Given n decimal digits $d_1, d_2, \ldots, d_n$ of $x \in [0, 1]$

$$x = 0.d_1 d_2 \ldots \in [0, 1]$$

- let $L_n(x)$ be the number of continued fraction digits (partial quotients) that are determined

$$x = [0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

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It is natural to consider the quotient $L_n(x)/n$

- This is the rate of continued fraction digits per decimal digit
- It allows the comparison of relative information of expansions

Continued fractions vs. decimal expansions and entropy

Let x_n, y_n with $x_n < x < y_n$ be the two consecutive n-th decimal approximations of x in $[0,1]$

Let $L_n(x)$ be the largest integer $k \geq 0$ such that

$$x_n = [a_0; a_1, \cdots, a_k, \cdots] \qquad y_n = [a_0; a_1, \cdots, a_k, \cdots]$$

have the same k first partial quotients

Theorem [Lochs'64] For almost every irrational number x

$$\lim \frac{L_n(x)}{n} = \frac{6 \log 10 \log 2}{\pi^2} \sim 0.9702$$

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- "The n first decimals determine the n first partial quotients"
- The first 1000 decimals of $\pi - 3$ give the first 968 partial quotients
- The continued fraction is only slightly more efficient at representing real numbers than the decimal expansion

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Theorem [Lochs'64] For almost every irrational number x

$$\lim \frac{L_n(x)}{n} = \frac{6 \log 10 \log 2}{\pi^2} \sim 0.9702$$

The quantity $h_G = \frac{\pi^2}{6 \log 2}$ is the entropy of the Gauss map $T_G : x \mapsto 1/x \mod 1$

The quantity $h_{10} = \log 10$ is the entropy of the map $T_{10} : x \mapsto 10x \mod 1$

$$\lim \frac{L_n(x)}{n} = h_{10}/h_G$$

Some improvements of Lochs' theorem

Theorem [Faivre] Let $x \in (0,1)$ be such that

- the growth of its partial quotients satisfies $a_n(x) = o(\alpha^n)$, for all $\alpha > 1$
- $\lim_{n\to\infty} q_n(x)/n$ exists
- Let $\beta(x) := \lim_{n\to\infty} q_n(x)/n$ Lévy's constant

Then

$$\lim_{n\to\infty} \frac{L_n(x)}{n} = \frac{\log 10}{2\beta(x)}$$

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- As $\beta(x)$ takes arbitrarily large values, $L_n(x)/n$ might take arbitrarily small values
- The continued fraction expansion of e is given by

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \cdots]$$

$$\lim_{n} \frac{L_n(e)}{n} = 0$$

Some improvements of Lochs' theorem

Theorem [Faivre] Loi gaussienne

$$\mathrm{Leb}\left\{x \in [0,1] : \frac{L_n(x) - nh_{10}/h_G}{\sigma\sqrt{n}} \leq \theta\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-u^2/2}\, du$$

with $\sigma > 0$

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with $\sigma > 0$
Let us write x in base β with $\beta > 1$

$$x = \sum_{n \geq 0} x_n \beta^{-n} \qquad T_\beta \colon x \mapsto \{\beta x\} = \beta x \mod 1$$

Theorem [Barreira-Godofredo'08] For almost every irrational number x

$$\lim_{n\to\infty} \frac{L_n(x)}{n} = \frac{h_\beta(x)}{h_G(x)} = \frac{6 \log 2 \log \beta}{\pi^2}$$

Continued fractions and dynamical systems

Consider the Gauß map

$$T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\} = 1/x - [1/x]$$

$$x_1 = T(x) = \{1/x\} = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} - a_1$$

$$x = \frac{1}{a_1 + x_1}$$

$$a_n = \left[\frac{1}{T^{n-1}x}\right]$$

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

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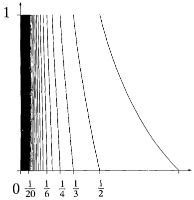


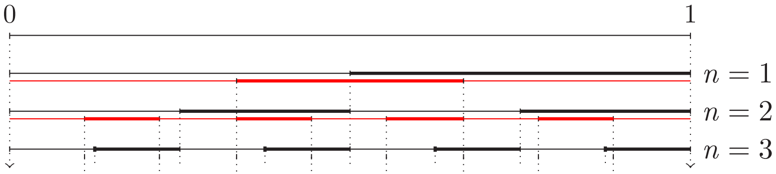
Outline

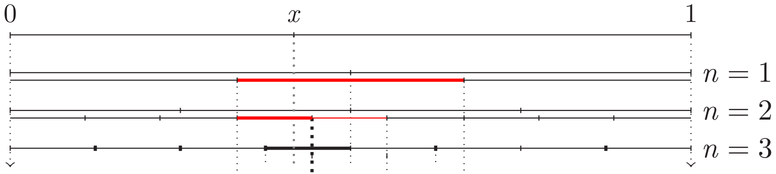
How large is the number $L_n(x)$ of digits determined in one expansion of a real number $x \in (0, 1)$ when a number n of digits of x are given in some other expansion?

- Lochs' index stated in terms of partitions
- Our extension to zero/infinite entropy
- Multidimensional case

Lochs' index

In black, the sequence of partitions $\mathcal{B}$ associated with base 2
In red, the sequence of partitions $\mathcal{T}$ associated with base 3





$$I_3^{\mathcal{B}}(x) \subseteq I_1^{\mathcal{T}}(x) \quad \text{but} \quad I_3^{\mathcal{B}}(x) \not\subseteq I_2^{\mathcal{T}}(x)$$

The first 3 binary digits of x only provide 1 ternary digit

Partitions

- A topological partition of $[0, 1]$ is a set P of intervals
  - open (nonempty)
  - disjoint
  - the union of their closures equals $[0, 1]$
- A sequence of partitions $\mathcal{P} = (P_n)_n$ is a sequence of topological partitions
- E is the set of endpoints of all the intervals of $\mathcal{P}$
- $\|\mathcal{P}_n\| = \sup\{|I| : I \in \mathcal{P}_n\}$ tends to 0

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- $I_n(x)$ is the only interval of P_n that contains x (if $x \notin E$)
- The first n symbols of $x \in [0,1]$ determine $I_n^{\mathcal{P}}(x)$ and conversely

Lochs' index

Consider

- $\mathcal{P}^1$ and $\mathcal{P}^2$ two sequences of partitions
- the interval $I_n^1(x)$ of depth n of $\mathcal{P}^1$ that contains x
- the interval $I_n^2(x)$ of depth n of $\mathcal{P}^2$ that contains x

Lochs' index

Consider

- $\mathcal{P}^1$ and $\mathcal{P}^2$ two sequences of partitions
- the interval $I_n^1(x)$ of depth n of $\mathcal{P}^1$ that contains x
- the interval $I_n^2(x)$ of depth n of $\mathcal{P}^2$ that contains x

Intuition

- If $I_n^1(x) \subset I_m^2(x)$, knowing $I_n^1(x)$ determines $I_m^2(x)$
- If $I_n^1(x) \not\subset I_m^2(x)$, then $I_n^1(x)$ might intersect several $J \in \mathcal{P}_m^2$
  $\rightsquigarrow$ we cannot decide which interval of $\mathcal{P}_m^2$ is $I_m^2(x)$

Lochs' index

Consider

- $\mathcal{P}^1$ and $\mathcal{P}^2$ two sequences of partitions
- the interval $I_n^1(x)$ of depth n of $\mathcal{P}^1$ that contains x
- the interval $I_n^2(x)$ of depth n of $\mathcal{P}^2$ that contains x

Lochs' index For $x \in [0,1]$ and each $n \in \mathbb{N}$, the Lochs's index is defined as

$$L_n(x;\mathcal{P}^1,\mathcal{P}^2) = \sup\{\ell \geq 0 : I_n^1(x) \subseteq I_\ell^2(x)\}$$

n digits of x in $\mathcal{P}^1$ provide $L_n(x;\mathcal{P}^1,\mathcal{P}^2)$ digits of x in $\mathcal{P}^2$

[Bosma-Dajani-Kraaincamp, Dajani-Fieldsteel]

On the entropy of the Gauss map

$$h_G = 2 \lim_n \frac{1}{n} \log q_n(x) = \frac{\pi^2}{6 \log 2}$$

Let

$$p_n/q_n = [0; a_1, \cdots, a_n]$$

The intervals $I_n(x)$ on which the partial quotients $a_1(x), \cdots, a_n(x)$ are fixed have the form $[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}]$ (or $[\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}]$ according to the parity of n) and satisfy

$$|I_n(x)| = \frac{1}{q_n(q_n + q_{n-1})}$$

Theorem [Khintchin-Lévy] $\lim_n \frac{1}{n} \log q_n(x) = \frac{\pi^2}{12 \log 2}$ a.e.

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[Shannon-McMillan-Breiman's Theorem] $h_G = \lim_{n\to\infty} \frac{-\log |I_n(x)|}{n}$ a.e.

Entropy of a sequence of partitions $\mathcal{P}$

Let $\mathcal{P} = (P_n)_n$ be a sequence of partitions
We assume that the set E of endpoints has zero measure

Definition The entropy of the sequence of partitions $\mathcal{P}$ is defined as

$$h(\mathcal{P}) = \lim_{n\to\infty} \frac{-\log|I_n(x)|}{n} \quad \text{a.e.}$$

if the limit exists

Example For the Gauss map, apply the Shannon-McMillan-Breiman Theorem

$$h(\mathcal{P}) = -\lim_{n} \frac{1}{n} \sum_{I\in P_n} |I| \log|I| = \lim_{n\to\infty} \frac{-\log|I_n(x)|}{n}$$

Lochs' index for positive entropy

Theorem [Dajani-Fieldsteel] Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions of $[0,1]$ of respective entropies $h(\mathcal{P}^1)$ and $h(\mathcal{P}^2)$
Then

$$\lim_{n\to\infty} \frac{1}{n} L_n(x;\mathcal{P}^1,\mathcal{P}^2) = \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)} \quad \text{a.e.}$$

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Lochs' theorem

- Decimals have a.e. entropy equal to $\log 10$
- Continued fractions have a.e. entropy equal to $\frac{\pi^2}{6\log 2}$

Beyond positive entropy

$$\lim_{n\to\infty} \frac{1}{n} L_n(x; \mathcal{P}^1, \mathcal{P}^2) = \begin{cases} 0 & h(\mathcal{P}^1) = 0 \\ \infty & h(\mathcal{P}^2) = 0 \end{cases}$$

Is it possible to be more precise in the case of zero entropy?

Weight and entropy

Let $\mathcal{P} = (P_n)_n$ be a sequence of partitions

Entropy

$$h(\mathcal{P}) = \lim_{n\to\infty} \frac{-\log|I_n(x)|}{n} \qquad \text{a.e.}$$

if the limit exists

Weight A map $f : \mathbb{N} \to R$ such that

$$\lim_{n\to\infty} \frac{-\log|I_n(x)|}{f(n)} = 1 \qquad \text{a.e.}$$

$$-\log|I_n^{\mathcal{P}}(x)| \sim f(n) \qquad \text{a.e.}$$

Weight for positive entropy

$$f(n) = h(\mathcal{P})n$$

Weight functions allow a generalization of the entropy

Log-balanced sequences of partitions and weight functions

Let $\mathcal{P} = (P_n)_n$ be a sequence of partitions
We assume that the set E of endpoints has zero measure

Definition $\mathcal{P}$ is log-balanced a.e. if there is some function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{-\log |I_n(x)|}{f(n)} = 1 \quad \text{a.e.}$$

If so, f is called a weight function of $\mathcal{P}$ a.e.

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A realization result Given any $f : \mathbb{N} \to \mathbb{R}$, $f(n) \to \infty$ as $n \to \infty$, there exists a sequence of partitions that has f as an a.e. weight function

Beyond positive entropy

Theorem Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions
The following limit holds

$$\lim_{n\to\infty} \frac{f_2(L_n(x,\mathcal{P}^1,\mathcal{P}^2))}{f_1(n)} = 1 \quad \text{a.e.}$$

if

- f_1 and f_2 are the corresponding weight functions
- $\lim_{n\to\infty} f_1(n)/\log n = +\infty$
- f_2 is nondecreasing
- $\sqrt[n]{|f_2(n)|} \to 1$ as $n \to \infty$

Idea of the proof

$$L_n(x;\mathcal{P}^1,\mathcal{P}^2) = \sup\{\ell \geq 0 : I_n^1(x) \subseteq I_\ell^2(x)\}$$

For a log-balanced sequence of partitions with weight function f

$$|I_n(x)| \approx e^{-f(n)}$$

Roughly

$$L_n(x;\mathcal{P}_1,\mathcal{P}_2) = m \quad \text{means} \quad |I_n^1(x)| \approx |I_m^2(x)|$$

Then

$$e^{-f_1(n)} \approx |I_n^1(x)| \approx |I_m^2(x)| \approx e^{-f_2(m)}$$

So

$$L_n(x;\mathcal{P}_1,\mathcal{P}_2) = m \approx f_2^{-1}(f_1(n))$$

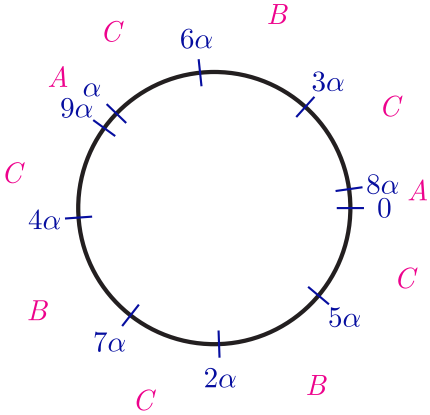
Finally

$$\frac{f_2(L_n(x;\mathcal{P}_1,\mathcal{P}_2))}{f_1(n)} \approx 1$$

Three-distance theorem

Let α be given and let us place the points $0, \alpha, 2\alpha, \cdots, N\alpha$ on the unit circle

Theorem The points $0, \alpha, 2\alpha, \cdots, N\alpha$ partition the unit circle into intervals having at most three lengths, one being the sum of the other two



Three-distance theorem

Let α be given and let us place the points $0, \alpha, 2\alpha, \cdots, N\alpha$ on the unit circle

It is also called the Steinhaus theorem, the three length, the three gap, or else, the three step theorem

It was initially conjectured by Steinhaus, first proved by V.T. Sós'58, Surányi'58, Slater'64, Świerczkowski'59, Halton'65

It is related to the table maker's dilemma

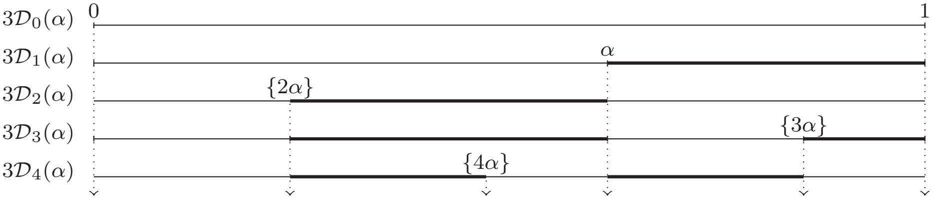
There exist various types of proofs: arithmetical, combinatorial, dynamical, geometry of the space of $2d$ lattices, etc.

The three distance partition $3\mathcal{D}(\alpha)$

Fix an irrational $\alpha \in (0,1)$

Consider the sequence $\{k\alpha\}_k = k\alpha \mod 1$

The intervals of $3\mathcal{D}_\alpha(n)$ have the points $\{k\alpha\}_{1\leq k\leq n}$ as endpoints



The three distance partition

Three-distance sequence of partitions The sequence of partitions $3\mathcal{D}(\alpha)$ is

- log-balanced a.e. with weight function

$$f(n) = \log n$$

  for α in a set of measure 1

- There exists an uncountable set of α's for which the sequence of partitions $3\mathcal{D}(\alpha)$ is not log-balanced

The proof is based on the three-distance theorem

Ostrowski's numeration

- The Ostrowski's map is defined as $S(x, y) = (\{1/x\}, \{y/x\})$
- For $(x, y) \in [0, 1]^2$, set

$$(x_0, y_0) := (x, y), \quad (x_i, y_i) := S^i(x, y) \quad \text{for all } i \geq 1$$

- We get a sequence of (pairs of) digits

$$(a_i, b_i) = \left(\left\lfloor \frac{1}{x_{i-1}} \right\rfloor, \left\lfloor \frac{y_{i-1}}{x_{i-1}} \right\rfloor \right)$$

- The sequence $(a_i)_i$ provides the continued fraction expansion of $x = [0; a_1, a_2, \cdots]$.

$$x_1 = 1/x - a_1 \rightsquigarrow x = \frac{1}{a_1 + x_1}$$

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- The sequence $(a_i)_i$ provides the continued fraction expansion of $x = [0; a_1, a_2, \cdots]$.
- Set

$$\theta_i := q_i x - p_i, \quad \text{with } p_i/q_i = [0; a_1, \cdots, a_i].$$

The sequence $(b_i)_i$ yields the digits for the Ostrowski representation of y w.r.t. the irrational base x

$$y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}| \qquad N = \sum_{i=1}^{\infty} b_i q_{i-1} \qquad \text{numeration system}$$

Ostrowski's numeration

Consider the second coordinate

$$S(x, y) = (\{1/x\}, \{y/x\}) = (1/x - a, y/x - b)$$

$$x_1 = 1/x - a_1 \rightsquigarrow x = \frac{1}{a_1 + x_1}$$

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$$x_1 = 1/x - a_1 \rightsquigarrow x = \frac{1}{a_1 + x_1}$$

$$y_1 = y/x - b_1 \rightsquigarrow y = x(b_1 + y_1)$$

$$y_{i-1} = x_{i-1}(b_i + y_i)$$

$$y = \sum_{i=1}^{n} b_i x_0 x_1 \cdots x_{i-1} + x_0 x_1 \cdots x_{n-1} y_n.$$

We then use the identity $x_0 x_1 \cdots x_i = (-1)^i \theta_i = |q_i x - p_i|$

$$\rightsquigarrow y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}|$$

Multidimensional case

Theorem [Dajani-De Vries-Johnson] Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$ of the square $[0,1]^2$ satisfying

(i) $\mathcal{P}^1$ is made out of squares

(ii) $\mathcal{P}^2$ is made out of convex polygons of entropy $h(\mathcal{P}^2) > 0$

(iii) There are constants $\beta, c_0, c_1 > 0$ such that for every I in every partition in $\mathcal{P}^2$

$$c_0 \lambda(I) \leq (\mathrm{diam}(I))^\beta \leq c_1 \lambda(I)$$

Then

$$\lim_{n\to\infty} \frac{1}{n} L_n(v; \mathcal{P}^1, \mathcal{P}^2) = \frac{\beta}{2(\beta-1)} \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)} = \frac{h_{\mathcal{P}^1}}{2\gamma h_{\mathcal{P}^2}} \quad \text{a.e.}$$

$$\gamma = 1 - 1/\beta$$

γ-geometric partitions

We relax the condition

$$c_0 \lambda(I) \leq (\operatorname{diam}(I))^{\beta} \leq c_1 \lambda(I)$$

γ-geometric partitions

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Definition

Given $\gamma < 1$, a system of partitions $\mathcal{P}$ of $[0,1]^2$ is γ-geometric if

$$\frac{\log \mathrm{diam}(I_k(v))}{\log \lambda(I_k(v))} \to 1-\gamma \quad \text{a.e.}$$

γ-geometric partitions

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Given $\gamma < 1$, a system of partitions $\mathcal{P}$ of $[0,1]^2$ is γ-geometric if

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$$\gamma = 1 - 1/\beta\,, \quad \beta = 1/(1-\gamma)$$

γ-geometric partition

Does the statement from **Dajani&DeVries&Johnson'05** read

$$\lim \tfrac{1}{n} L_n(v; \mathcal{P}^1, \mathcal{P}^2) = \frac{\gamma_1}{\gamma_2} \times \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}$$

where $\gamma_1 = 1/2$ for squares,

γ-geometric partition

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$$\lim \tfrac{1}{n} L_n(v; \mathcal{P}^1, \mathcal{P}^2) = \frac{\gamma_1}{\gamma_2} \times \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}$$

where $\gamma_1 = 1/2$ for squares, or is it rather

$$\lim \tfrac{1}{n} L_n(v; \mathcal{P}^1, \mathcal{P}^2) = \frac{1-\gamma_1}{\gamma_2} \times \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)} \ ?$$

γ-geometric partition

Does the statement from **Dajani&DeVries&Johnson'05** read

$$\lim \tfrac{1}{n} L_n(v; \mathcal{P}^1, \mathcal{P}^2) = \frac{\gamma_1}{\gamma_2} \times \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}$$

where $\gamma_1 = 1/2$ for squares, or is it rather

$$\lim \tfrac{1}{n} L_n(v; \mathcal{P}^1, \mathcal{P}^2) = \frac{1-\gamma_1}{\gamma_2} \times \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)} \ ?$$

We show that both limits can be attained for $\gamma_1 \neq 1/2$

$$\frac{\log \operatorname{diam}(I_k(v))}{\log \lambda(I_k(v))} \to 1 - \gamma \quad \text{a.e.}$$

Theorem Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions of $[0,1]^2$, which are respectively γ_1 and γ_2-geometric, with positive entropy h_1 and h_2
Under mild geometric conditions

$$\frac{1-\gamma_1}{\gamma_2}\frac{h_1}{h_2} \le \liminf_{n\to\infty} \frac{L_n(v;\mathcal{P}^1,\mathcal{P}^2))}{n} \le \limsup_{n\to\infty} \frac{L_n(v;\mathcal{P}^1,\mathcal{P}^2))}{n} \le \min\left(\frac{\gamma_1}{\gamma_2}, \frac{1-\gamma_1}{1-\gamma_2}\right)\frac{h_1}{h_2}$$